DPG Approach for Dealing with Stress Concentrations

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How to deal with stress?

Photo: Esko Sistonen

Stress singularities and concentrations are common in structural engineering

One can either

1. **Ignore them (St. Venant's principle)**
2. **Resolve them by refined analysis**

“The exact distribution of a load is not important far away from the loaded region, as long as the resultants of the load are correct” [Saint-Venant, 1855]

Causes of stress singularities/concentrations:

- a concentrated load
- abrupt local transitions in loading
- constraining a model at a point
- abrupt transitions in kinematic constraints
- abrupt transition between materials
- sharp re-entrant corners

Curved shell structures may feature unique parameter-dependent stress concentrations

Standard DPG setup

(for stress relieve)

Standard DPG setup

Petrov-Galerkin approximation:

$$\mathbf{u}_h \in U_h \subset U : \qquad b(\mathbf{u}_h, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in T(U_h)$$

with trial-to-test operator T:

$$T : U \to V : \quad \langle\!\langle T\mathbf{u}, \mathbf{v} \rangle\!\rangle_V = b(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in V$$

is inf-sup stable and converges optimally:

$$\|\mathbf{u} - \mathbf{u}_h\|_E = \min\{\|\mathbf{u} - \mathbf{w}\|_E;\ \mathbf{w} \in U_h\} \quad \text{where} \quad \|\mathbf{w}\|_E := \|B\mathbf{w}\|_{V'}.$$

L. Demkowicz, J. Gopalakrishnan, Comput. Methods Appl. Mech. Engrg. 2010

Standard DPG setup

- Ultraweak formulation
- Discontinuous test spaces
- Independent trace variables
- Optimal test functions

DPG method: properties

- Continuous stability implies discrete stability.
- Discrete system is SPD:

$$u_h \in U_h : \quad \|B(u-u_h)\|_{V'} \to \min \quad \text{minimum residual}$$

- Method provides best approximation:

$$\|u-u_h\|_E = \inf_{w_h \in U_h} \|u-w_h\|_E, \quad \|u\|_E := \|Bu\|_{V'} \quad \text{residual “energy” norm}$$

- The energy norm of the error can be evaluated by solving

$$\psi \in V : \ (\psi, v)_V = b(u-u_h, v) = L(v) - b(u_h, v) \quad \forall v \in V$$

$$\Rightarrow \quad \|u-u_h\|_E = \|\psi\|_V \ \ldots \ = \Big(\sum_T \|\psi_T\|^2_{V(T)}\Big)^{1/2}$$

DPG theory (i)

Optimal test norm

$$\|v\|_{V,\mathrm{opt}} := \sup_{u \neq 0} \frac{b(u,v)}{\|u\|_U}$$

is impractical: global, coupled variables.
The aim is to employ a decoupled, localizable norm $\|v\|_V$.

Proving the norm equivalence

$$\|v\|_V \lesssim \|v\|_{V,\mathrm{opt}} \tag{1}$$

yields

$$\|u\|_U = \sup_{v \neq 0} \frac{b(u,v)}{\|v\|_{V,\mathrm{opt}}} \lesssim \sup_{v \neq 0} \frac{b(u,v)}{\|v\|_V} =: \|u\|_E$$

with corresponding error estimate.

Bound (1) implies robustness of the method
and is equivalent to the stability of the adjoint problem.

DPG theory (ii)

Ultraweak formulation with field variable(s) u and trace(s) $\hat{u}$:

$$(u, \hat{u}) \in U_0 \times \widehat{U}: \quad b((u, \hat{u}), v) = L(v) \quad \forall v \in V$$

$$V_0 := \{v \in V;\ b((0, \hat{u}), v) = 0\ \forall \hat{u} \in \widehat{U}\} \qquad \text{“}[v] = 0\text{”}$$

Proposition [Carstensen, Gopalakrishnan, Demkowicz ’16]
$b(\cdot, \cdot)$ satisfies inf-sup property if

$$\|v\|_V \lesssim \|v\|_{V,\mathrm{opt}} \quad \forall v \in V_0$$

(stability of continuous adjoint problem), and

$$\sup_{\|v\|_V = 1} b((0, \hat{u}), v) \gtrsim \|\hat{u}\|_{\widehat{U}} \quad \forall \hat{u} \in \widehat{U}$$

(boundedness below of trace operator).

Kirchhoff-Love model

DPG for Kirchhoff-Love model

$$
\begin{aligned}
\operatorname{div} \mathbf{div}\, \mathbf{M} &= f \quad \text{in} \quad \Omega \\
\mathbf{M} - \varepsilon \nabla u &= 0 \quad \text{in} \quad \Omega \\
u = 0, \quad \partial_n u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

Testing with p/w smooth $z, \mathbf{Q}$:

$$
(\mathbf{M}, \varepsilon_{\mathcal{T}} \nabla_{\mathcal{T}} z) + \sum_{T \in \mathcal{T}} \langle n \cdot \mathbf{div}\, \mathbf{M}, z \rangle_{\partial T} - \sum_{T \in \mathcal{T}} \langle \mathbf{M} n, \nabla z \rangle_{\partial T}
$$

$$
+ (\mathbf{M}, \mathbf{Q}) - (u, \operatorname{div}_{\mathcal{T}} \mathbf{div}_{\mathcal{T}} \mathbf{Q}) - \sum_{T \in \mathcal{T}} \langle \mathbf{Q} n, \nabla u \rangle_{\partial T} + \sum_{T \in \mathcal{T}} \langle n \cdot \mathbf{div}\, \mathbf{Q}, u \rangle_{\partial T} = (f, z)
$$

Discretization of the traces

Lowest order discrete spaces

$$U_{\mathrm{dDiv},T} := \{\mathbf{Q} \in \mathcal{H}(\mathrm{div}\,\mathbf{div}\,, T);\ \varepsilon\nabla\,\mathrm{div}\,\mathbf{div}\,\mathbf{Q} + \mathbf{Q} = 0,$$
$$\big(n\cdot\mathbf{div}\,\mathbf{Q} + \partial_{\mathbf{t},\mathcal{E}_T}(n\cdot\mathbf{Qt})\big)|_{\partial T} \in P^0(\mathcal{E}_T),\quad n\cdot\mathbf{Q}n|_{\partial T} \in P^0(\mathcal{E}_T)\}$$

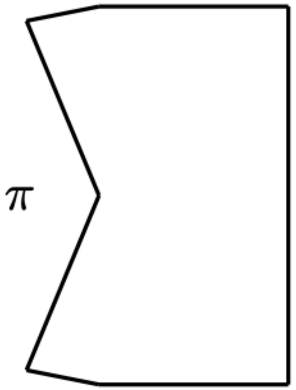
Degrees of freedom of $\widehat{Q}_S = \mathrm{tr}^{\mathrm{dDiv}}(U_{\mathrm{dDiv},\mathcal{T}})$ are

$$\begin{aligned} &\langle n\cdot\mathbf{div}\,\mathbf{Q} + \partial_{\mathbf{t}}(n\cdot\mathbf{Qt})\,,1\rangle_E && (E \text{ edge}),\\ &\langle n\cdot\mathbf{Q}n, 1\rangle_E && (E \text{ edge}),\\ &[\![\mathbf{Q}]\!]_{\partial T}(e) && (e \text{ vertex of } T)\\ \text{subject to } &\sum_{T\in\omega(e)} [\![\mathbf{Q}]\!]_{\partial T}(e) = 0 && \forall \text{interior vertex } e. \end{aligned}$$

These are two constants on each edge and deltas at vertices.

Kirchhoff-Love: numerical example

$$u(r,\varphi) = r^{1+\alpha}\big(\cos((\alpha+1)\varphi) + C\cos((\alpha-1)\varphi)\big)$$
$$\operatorname{div}\mathbf{div}\,\varepsilon\nabla u = 0, \qquad \alpha \approx 0.67, \qquad C \approx 1.23$$

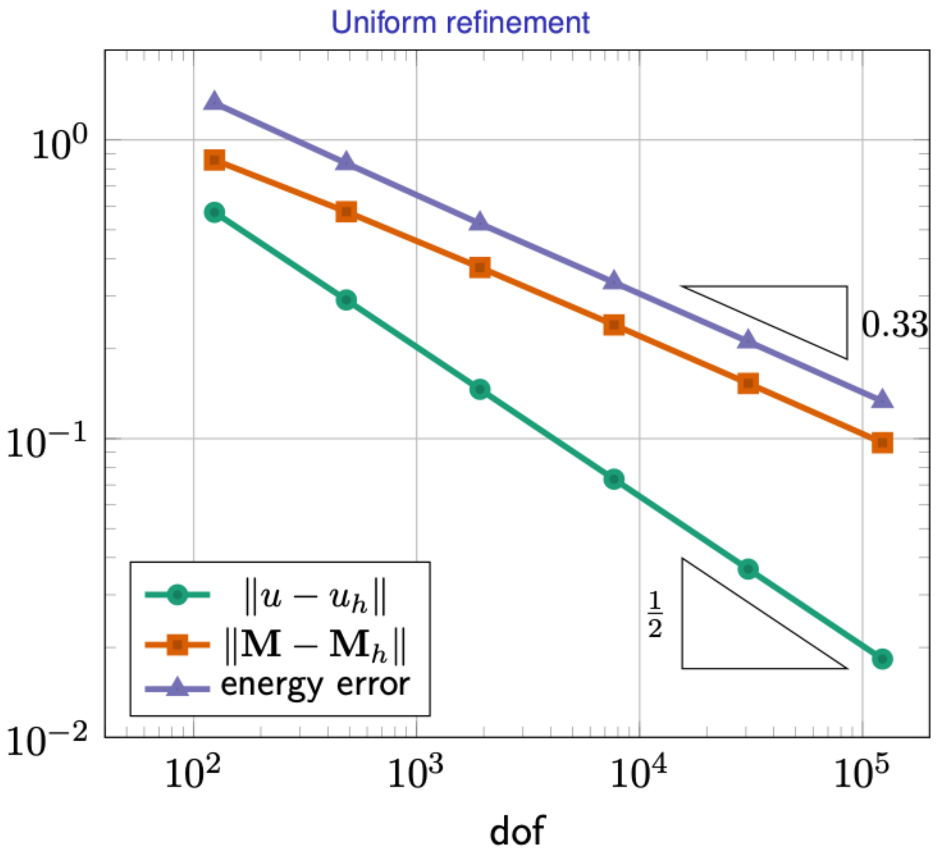


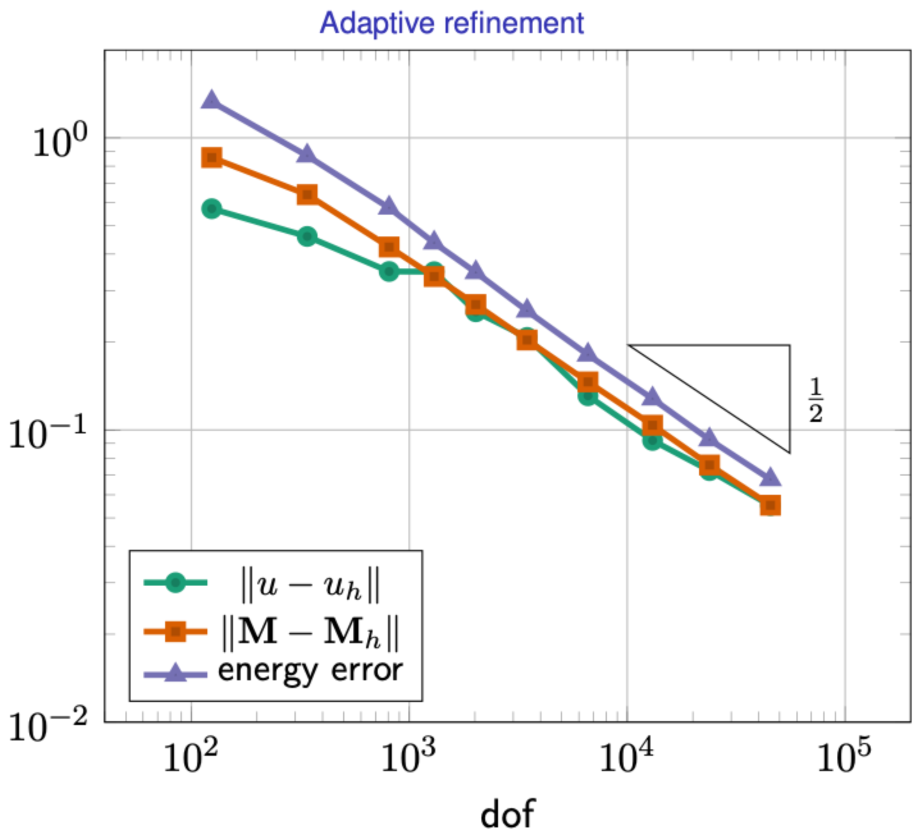
$$\begin{aligned} u \quad & \in H^{2+\alpha-\varepsilon}(\Omega) \\ \mathbf{M} = \varepsilon\nabla u & \in (H^{\alpha-\varepsilon}(\Omega))^{2\times 2} \\ \mathbf{div}\,\mathbf{M} \quad & \notin L_2(\Omega) \\ \mathbf{M} \quad & \in \mathcal{H}(\operatorname{div}\mathbf{div},\Omega) \end{aligned}$$

$\mathcal{U}_H$: p/w constants for $u, \mathbf{M}, \mathrm{tr}^{\mathrm{dDiv}}(\mathbf{M})$, lowest order HCT for $\mathrm{tr}^{\mathrm{Ggrad}}(u)$.

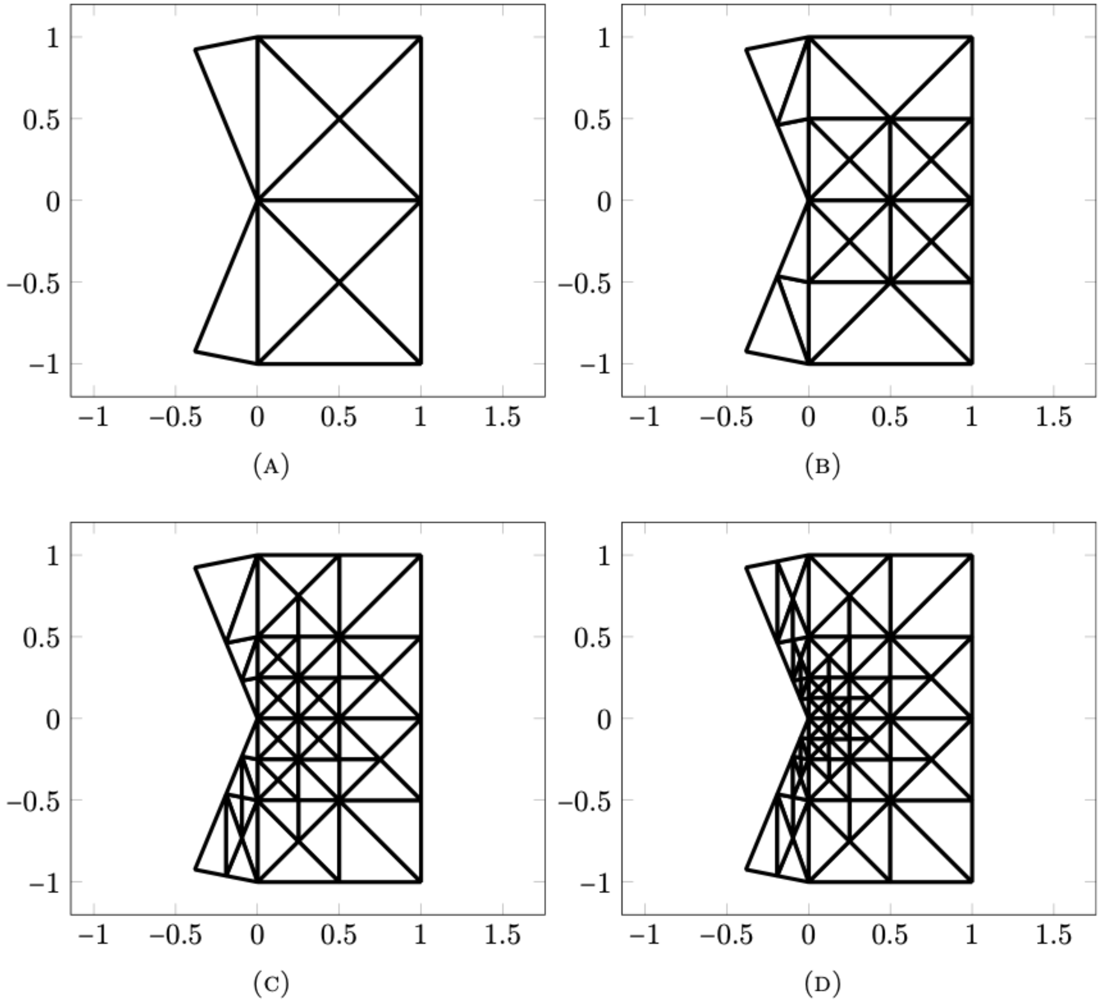
Expected order: $O(h^\alpha) = O(\dim(\widehat{\mathcal{U}}_h)^{-\alpha/2}) \approx O(\dim(\widehat{\mathcal{U}}_h)^{-0.33})$.

Convergence





Adaptive refinement



Shallow Koiter model

Shallow Koiter model

$$\mathbf{B} : \mathbf{N} - \operatorname{div} \mathbf{div}\, \mathbf{M} = f \quad (\Omega) \qquad f\text{: vertical load}$$

$$\mathbf{M} - \mathcal{C}_b \boldsymbol{\kappa} = 0 \quad (\Omega)$$

$$\mathbf{N} - \mathcal{C}_m \boldsymbol{\beta} = 0 \quad (\Omega)$$

$$-\mathbf{div}\, \mathbf{N} = \mathbf{p} \quad (\Omega) \qquad \mathbf{p}\text{: tangential load}$$

$$\boldsymbol{u} = 0,\ w = 0,\ \boldsymbol{n} \cdot \mathbf{M}\boldsymbol{n} = 0 \quad (\partial\Omega) \qquad \text{simply supported}$$

$$\boldsymbol{u} = 0,\ w = 0,\ \partial_{\boldsymbol{n}} w = 0 \quad (\partial\Omega) \qquad \text{clamped}$$

$\boldsymbol{u}$	tangential displacements,	w	transverse deflection
$\mathbf{M}$	bending moments,	$\boldsymbol{\kappa} = -\nabla^s \nabla w$	bending curvatures
$\mathbf{N}$	membrane forces,	$\boldsymbol{\beta} = \nabla^s \boldsymbol{u} + \mathbf{B} w$	membrane strains
$\mathbf{B}$	curvature tensor,	∇^s	symmetric gradient
$\mathcal{C}_m$	tensor $\sim d$: thickness,	$\mathcal{C}_b$	tensor $\sim d^3$

DPG formulation

$$\mathbf{B}:\mathbf{N} - \mathrm{div}\,\mathbf{div}\,\mathbf{M} = f \,|\, \cdot z, \qquad \boldsymbol{n}\cdot\mathbf{div}\,\mathbf{M}|_{\mathcal{S}}, \mathbf{M}\boldsymbol{n}|_{\mathcal{S}}$$

$$\mathbf{M} + d^2\nabla^s\nabla w = 0 \,|\, : d^{-2}\mathbf{S}, \qquad \nabla w|_{\mathcal{S}}, w|_{\mathcal{S}}$$

$$\mathbf{N} - (\nabla^s \boldsymbol{u} + \mathbf{B}w) = 0 \,|\, : \mathbf{T}, \,|\, : \mathbf{Q} \qquad \boldsymbol{u}|_{\mathcal{S}}$$

$$-\mathbf{div}\,\mathbf{N} = \mathbf{p} \,|\, \cdot \boldsymbol{v}, \qquad \mathbf{N}\boldsymbol{n}|_{\mathcal{S}}$$

Spaces with norms (with appropriate $c_Q > 0$ and tensor $\mathbf{C}_{\mathrm{disp}} > 0$)

$$U_0 := \boldsymbol{H}^1(\Omega) \times H^2(\Omega) \times \mathbb{H}(\mathbf{div}, \Omega) \times \mathbb{H}(\mathrm{div}\,\mathbf{div}, \Omega),$$

$$\|\mathbf{C}_{\mathrm{disp}}\boldsymbol{u}\|^2 + \|\nabla^s\boldsymbol{u} + \mathbf{B}w\|^2 + d^2\|w\|^2 + d^2\|\nabla^s\nabla w\|^2 + \|\mathbf{N}\|^2$$
$$+ c_Q^{-1}\|\mathrm{skew}(\mathbf{N})\|^2 + \|\mathbf{C}_{\mathrm{disp}}^{-1}\mathbf{div}\,\mathbf{N}\|^2 + d^{-2}\|\mathbf{M}\|^2 + d^{-2}\|\mathrm{div}\,\mathbf{div}\,\mathbf{M} - \mathbf{B}:\mathbf{N}\|^2,$$

$$V(\mathcal{T}) := \boldsymbol{H}^1(\mathcal{T}) \times H^2(\mathcal{T}) \times \mathbb{H}^s(\mathbf{div}, \mathcal{T}) \times \mathbb{H}(\mathrm{div}\,\mathbf{div}, \mathcal{T}) \times \mathbb{L}_2^k(\mathcal{T}),$$

$$\|\mathbf{C}_{\mathrm{disp}}\boldsymbol{v}\|_{\mathcal{T}}^2 + \|\nabla^s\boldsymbol{v} - \mathbf{B}z + \mathbf{Q}\|_{\mathcal{T}}^2 + d^2\|z\|_{\mathcal{T}}^2 + d^2\|\nabla^s\nabla z\|_{\mathcal{T}}^2 + \|\mathbf{T}\|_{\mathcal{T}}^2$$
$$+ \|\mathbf{C}_{\mathrm{disp}}^{-1}\mathbf{div}\,\mathbf{T}\|_{\mathcal{T}}^2 + d^{-2}\|\mathbf{S}\|_{\mathcal{T}}^2 + d^{-2}\|\mathrm{div}\,\mathbf{div}\,\mathbf{S} - \mathbf{B}:\mathbf{T}\|_{\mathcal{T}}^2 + c_Q\|\mathbf{Q}\|_{\mathcal{T}}^2$$

Numerical discretization

Approximation space U_h:

p/w constants on $\mathcal{T}$	for	$\boldsymbol{u}$, w, $\mathbf{N}$ (symm), $\mathbf{M}$
HCT-traces on $\mathcal{S}$	for	trace of w
KLove traces on $\mathcal{S}$	for	trace of $\mathbf{M}$
p/w constants on $\mathcal{S}$	for	normal trace of $\mathbf{N}$
continuous p/w linears/quadratics on $\mathcal{S}$	for	trace of $\boldsymbol{u}$

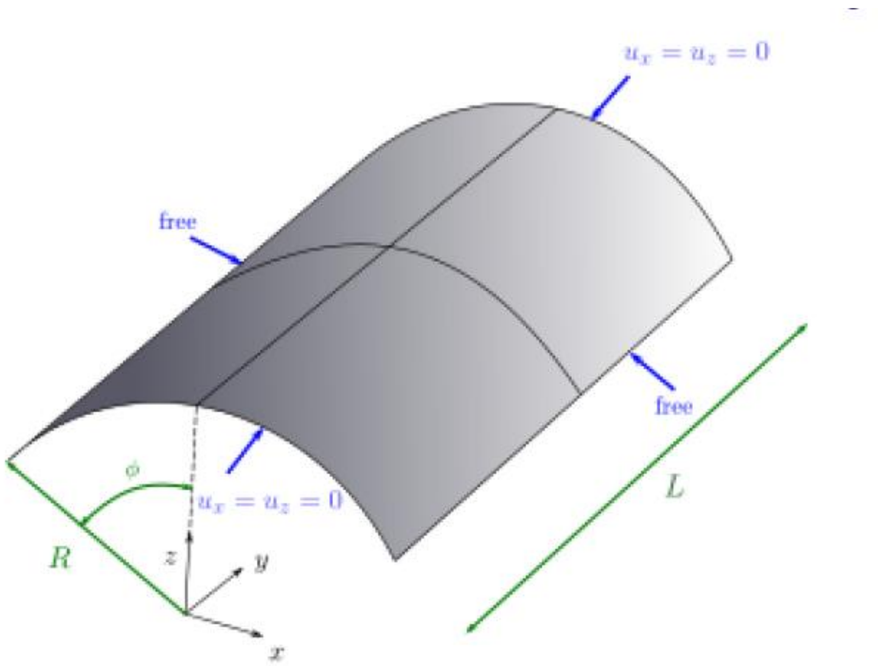
Test space

$$V(\mathcal{T}) = \boldsymbol{H}^1(\mathcal{T}) \times H^2(\mathcal{T}) \times \mathbb{H}^s(\mathbf{div}, \mathcal{T}) \times \mathbb{H}(\mathrm{div}\,\mathbf{div}, \mathcal{T}) \times \mathbb{L}_2^k(\mathcal{T})$$

replaced with discrete test space

$$[P^3(\mathcal{T})]^2 \times P^3(\mathcal{T}) \times [P^3(\mathcal{T})]^{2\times 2,\mathrm{sym}} \times [P^4(\mathcal{T})]^{2\times 2,\mathrm{sym}} \times \{0\}.$$

Scordelis-Lo roof



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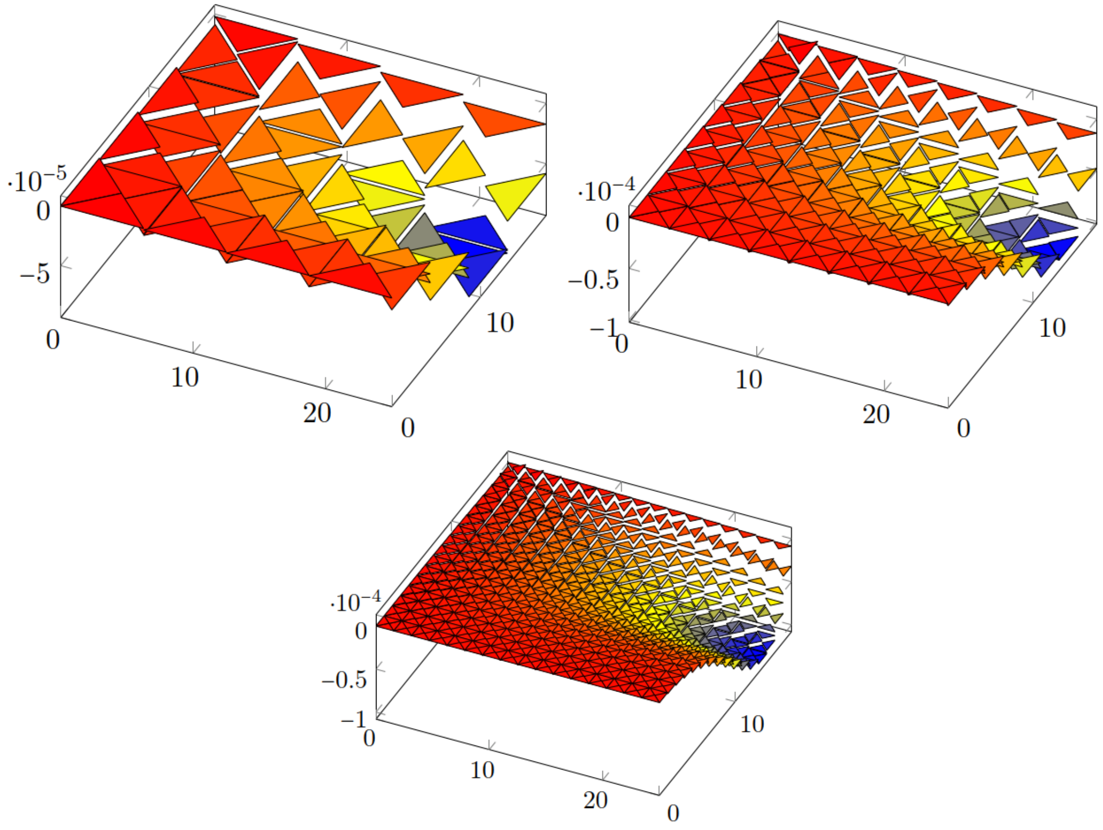
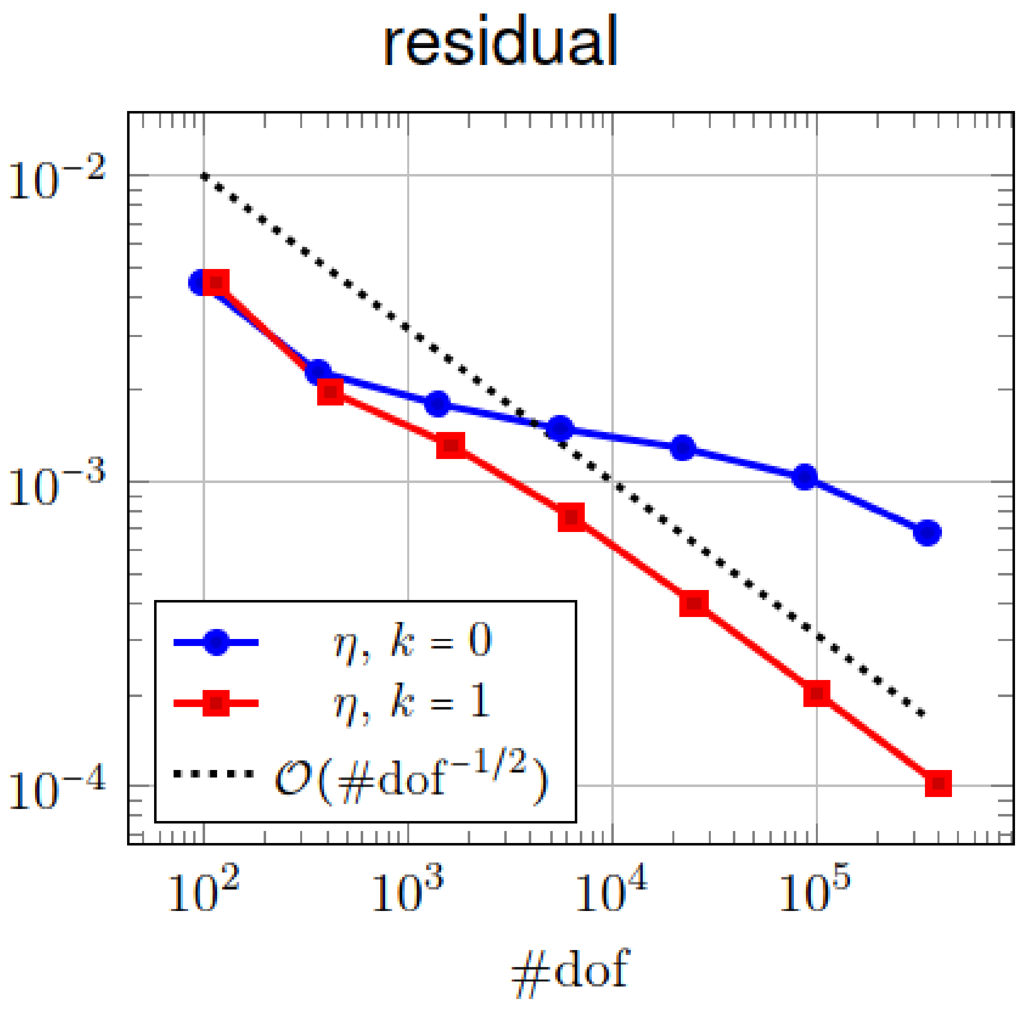
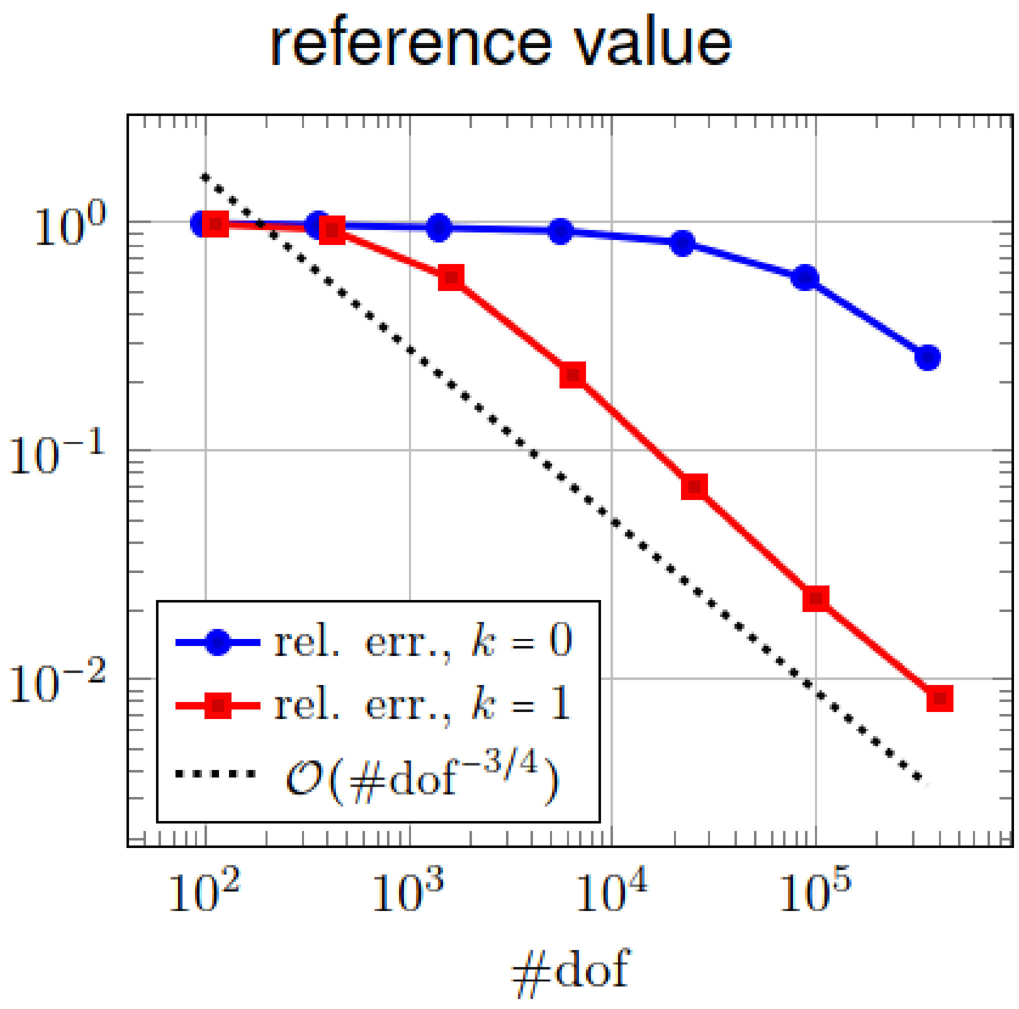
In-plane shear force N_{12} (mesh with 64, 256, 1024 elements)



Scordelis-Lo roof: convergence





Analysis of the ”hot spot”

$f = \delta_{(0,0)}$ at one node of an element, $\mathbf{p} = 0$

$$\Omega = (-1,1) \times (-1,1), \quad \mathbf{B}_{\text{ell}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B}_{\text{par}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B}_{\text{hyp}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$d = 10^{-2},\ E = 1,\ \nu = 0$

$\mathbf{C}_{\text{disp}} = \operatorname{diag}(d, d)$ (ell,par), $\quad \mathbf{C}_{\text{disp}} = \operatorname{diag}(1, 1)$ (hyp)

Appropriate homogeneous boundary conditions, Fourier solution

Stress concentration near the ”hot spot”

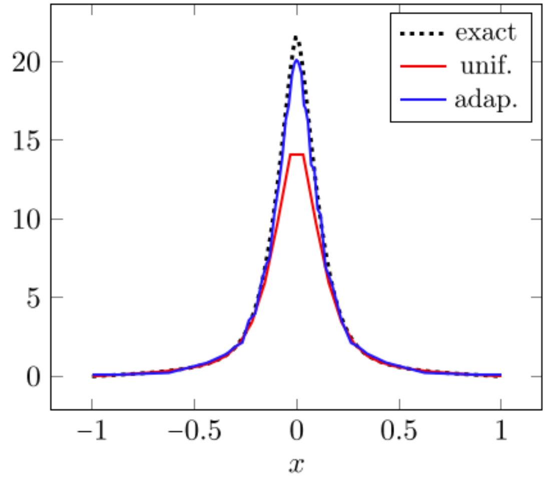


Fig. 8 Elliptic shell with point load, $d = 10^{-2}$, $k = 0$. Exact solution N_{11} along $y = 0$ and its approximations with uniform mesh (4096 triangles) and adaptively refined mesh (1828 triangles)

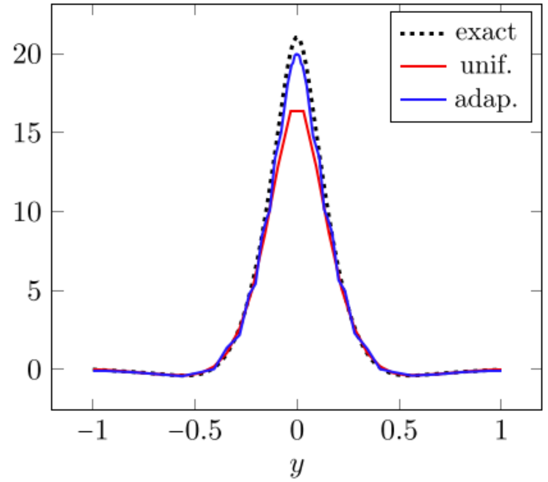
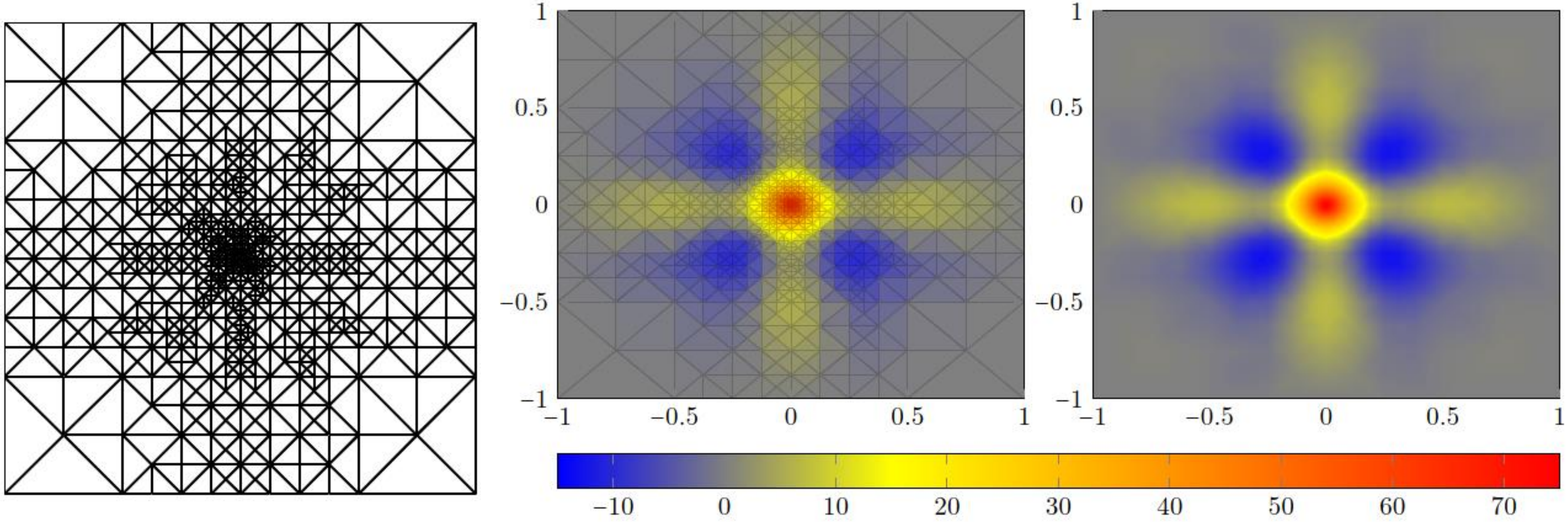


Fig. 10 Parabolic shell with point load, $d = 10^{-2}$, $k = 1$. Exact solution N_{22} along $x = 0$ and its approximations with uniform mesh (4096 triangles) and adaptively refined mesh (2139 triangles)

Hot spot on a hyperbolic shell at *R/t=100*

Hyperbolic case, $k = 1$

mesh (1294 elements), approximate & exact transverse deflections



Concluding remarks

- DPG provides stable numerical discretization of plate and shell models
- Adaptivity is built in and works from the start (coarse mesh)
- The method provides accurate predictions of both displacements and stresses including shear forces!
- Numerical locking effects can be alleviated by appropriate trace approximations

References and acknowledgements

Führer, T., Heuer, N. & Niemi, A.H. A DPG method for shallow shells. Numer. Math. 152, 67–99 (2022).

Führer, T., Heuer N., and Niemi A.H., An ultraweak formulation of the Kirchhoff–Love plate bending model and DPG approximation, Math. Comp., 88 (2019),1587–1619.

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