DPG for Reissner–Mindlin plates, Part 2

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DPG method

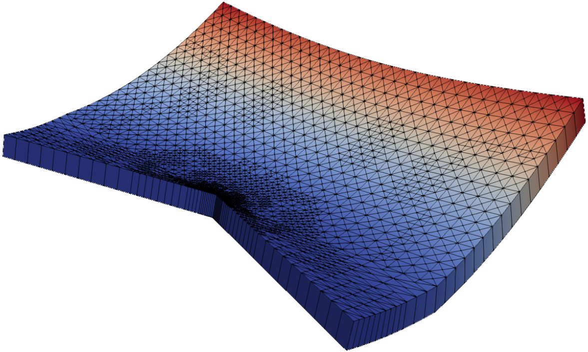
Framework by [Demkowicz, Gopalakrishnan '10]

- ultraweak formulation
- with product test spaces,
- independent trace variables, and
- optimal test functions.

Related ideas

- ultraweak formulation [Cessenat, Després '94, '98]
- independent trace variables [Botasso, Micheletti, Sacco '02],
- optimal test functions [Barret, Morton '84]
- mixed stabilization [Cohen, Dahmen, Welper '12], [Dahmen, Huang, Schwab, Welper '12]

Deformation of a thin elastic structure



Tofu design

Reissner–Mindlin model

$\boldsymbol{q}$ shear force vector, $\mathbf{M}$ symmetric bending tensor, f surface load,

$$-\operatorname{div}\boldsymbol{q} = f, \quad \boldsymbol{q} = \mathbf{div}\,\mathbf{M} \qquad \text{on } \Omega \quad \text{(midsurface)}$$

Linearly elastic isotropic material:

$$\mathbf{M} = -t^3\mathcal{C}\nabla^s\boldsymbol{\psi} = -Dt^3[\nu\,\mathrm{tr}\nabla^s\boldsymbol{\psi}\mathbf{I} + (1-\nu)\nabla^s\boldsymbol{\psi}]$$

$$\boldsymbol{q} = \kappa Gt(\nabla u - \boldsymbol{\psi}) \qquad \text{where}$$

$\boldsymbol{\psi}$ rotation vector, u deflection, $\nabla^s\boldsymbol{\psi} := (\nabla\boldsymbol{\psi} + \nabla\boldsymbol{\psi}^T)/2$,

$$G = \frac{E}{2(1+\nu)}, \quad D = \frac{E}{12(1-\nu^2)}, \quad \kappa > 0 \quad \text{(shear correction)},$$

E : Young modulus, ν : Poisson ratio, t : plate thickness.

Reissner–Mindlin model

Scaled problem: $f \to t^3 f$, $\mathbf{M} \to t^3\mathbf{M}$, $\kappa G = D = 1$, $\nu = 0$

$$
\begin{aligned}
-\operatorname{div} \mathbf{div}\, \mathbf{M} &= f \quad \text{in} \quad \Omega \\
\mathbf{M} + \mathcal{C}\nabla^s(\nabla u - t^2 \mathbf{div}\, \mathbf{M}) &= 0 \quad \text{in} \quad \Omega \\
u = 0, \quad (\psi =)\ \nabla u - t^2 \mathbf{div}\, \mathbf{M} &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

Formally setting $t = 0$: Kirchhoff–Love model

$$
\begin{aligned}
-\operatorname{div} \mathbf{div}\, \mathbf{M} &= f \quad \text{in} \quad \Omega \\
\mathbf{M} + \mathcal{C}\nabla^s\nabla u &= 0 \quad \text{in} \quad \Omega \\
u = 0, \quad \partial_n u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

In the following $\mathcal{C} = \mathbf{I}$.

Reissner–Mindlin, Part 1

$$
\begin{aligned}
\mathrm{div}(\mathbf{div}\,\mathbf{M} + t(\boldsymbol{\theta} - \nabla u)) &= -f && \text{in} \quad \Omega \\
\mathbf{M} + \nabla^s(\nabla u - t^2 \mathbf{div}\,\mathbf{M}) &= 0 && \text{in} \quad \Omega \\
t(\boldsymbol{\theta} - \nabla u) &= 0 && \text{in} \quad \Omega \\
u = 0, \quad \nabla u - t^2 \mathbf{div}\,\mathbf{M} &= 0 && \text{on} \quad \partial\Omega
\end{aligned}
$$

Testing with p/w smooth (mesh!) z, $\mathbf{Q}$ (symmetric tensors), and $\boldsymbol{\tau}$

$$
\begin{aligned}
(u, \mathrm{div}(\mathbf{div}\,\mathbf{Q} + t(\boldsymbol{\tau} - \nabla z)))_{\mathcal{T}} &+ (\mathbf{M}, \mathbf{Q} + \nabla^s(\nabla z - t^2 \mathbf{div}\,\mathbf{Q}))_{\mathcal{T}} \\
&+ t(\boldsymbol{\theta}, \boldsymbol{\tau} - \nabla z)_{\mathcal{T}} + \quad \text{“trace terms”} \quad = -(f, z)
\end{aligned}
$$

Reissner–Mindlin, Part 1

Traces

$$\langle \mathrm{tr}^{\mathrm{RM}}_{\mathcal{T},t}(u,\mathbf{M},\boldsymbol{\theta}),(z,\mathbf{Q},\boldsymbol{\tau})\rangle_{\mathcal{S},t} :=$$
$$\langle u, \boldsymbol{n}\cdot(\mathbf{div}\,\mathbf{Q}+t(\boldsymbol{\tau}-\nabla z))\rangle_{\mathcal{S}} - \langle \boldsymbol{n}\cdot(\mathbf{div}\,\mathbf{M}+t(\boldsymbol{\theta}-\nabla u)), z\rangle_{\mathcal{S}}$$
$$\langle \mathbf{M}\boldsymbol{n}, \nabla z - t^2\mathbf{div}\,\mathbf{Q}\rangle_{\mathcal{S}} - \langle \nabla u - t^2\mathbf{div}\,\mathbf{M}, \mathbf{Q}\boldsymbol{n}\rangle_{\mathcal{S}}$$

Regularities

u	$\in H^1_0(\Omega)$	$\Rightarrow\ \hat{u} := \mathrm{tr}(u)$	(std trace)
$\nabla u - t^2\mathbf{div}\,\mathbf{M} = \boldsymbol{\psi}$	$\in \boldsymbol{H}^1_0(\Omega)$	$\Rightarrow\ \hat{\boldsymbol{\psi}} := \mathrm{tr}(\boldsymbol{\psi})$	(std trace)
$\mathbf{M}$	$\in \mathbb{H}(\mathbf{div},\Omega)$	$\Rightarrow\ \hat{M} := \mathrm{tr}(\mathbf{M}\boldsymbol{n})$	(std trace)
$\mathbf{div}\,\mathbf{M}+t(\boldsymbol{\theta}-\nabla u)$	$\in \boldsymbol{H}(\mathrm{div},\Omega)$	$\Rightarrow\ \hat{q} := \mathrm{tr}(\boldsymbol{n}\cdot\mathbf{div}\,\mathbf{M})$	(std trace)

Introducing the independent variables $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{\psi}}$, $\hat{M}$, $\hat{q}$ we obtain a well-posed formulation

and a locking scheme, caused by $\mathrm{tr}(\nabla u - t^2\mathbf{div}\,\mathbf{M})$.

Reissner–Mindlin, Part 1

Traces

$$
\begin{aligned}
\langle \mathrm{tr}^{\mathrm{RM}}_{\mathcal{T},t}(u,\mathbf{M},\boldsymbol{\theta}),(z,\mathbf{Q},\boldsymbol{\tau})\rangle_{\mathcal{S},t} &= \\
&\langle u\,,\boldsymbol{n}\cdot(\mathbf{div}\,\mathbf{Q}+t(\boldsymbol{\tau}-\nabla z))\rangle_{\mathcal{S}} - \langle \boldsymbol{n}\cdot(\mathbf{div}\,\mathbf{M}+t(\boldsymbol{\theta}-\nabla u))\,,z\rangle_{\mathcal{S}} \\
&\langle \mathbf{M}\boldsymbol{n}\,,\nabla z - t^2\mathbf{div}\,\mathbf{Q}\rangle_{\mathcal{S}} - \langle \nabla u - t^2\mathbf{div}\,\mathbf{M}\,,\mathbf{Q}\boldsymbol{n}\rangle_{\mathcal{S}}
\end{aligned}
$$

$$
\begin{aligned}
&= (u\,,\mathrm{div}(\mathbf{div}\,\mathbf{Q}+t(\boldsymbol{\tau}-\nabla z)))_{\mathcal{T}} - (\mathrm{div}(\mathbf{div}\,\mathbf{M}+t(\boldsymbol{\theta}-\nabla u))\,,z) \\
&\quad +(\mathbf{M}\,,\nabla^s(\nabla z - t^2\mathbf{div}\,\mathbf{Q}))_{\mathcal{T}} - (\nabla^s(\nabla u - t^2\mathbf{div}\,\mathbf{M})\,,\mathbf{Q}) \\
&\quad -t(\boldsymbol{\theta}\,,\nabla z)_{\mathcal{T}} + t(\nabla u\,,\boldsymbol{\tau})
\end{aligned}
$$

motivates to use the following norms

$$
\begin{aligned}
\|(u,\mathbf{M},\boldsymbol{\theta})\|^2_{U(t)} &:= \|u\|^2 + t\|\nabla u\|^2 + \|\mathbf{M}\|^2 + t\|\boldsymbol{\theta}\|^2 \\
&\quad + \|\nabla^s(\nabla u - t^2\mathbf{div}\,\mathbf{M})\|^2 + \|\,\mathrm{div}(\mathbf{div}\,\mathbf{M}+t(\boldsymbol{\theta}-\nabla u))\|^2 \\
\|(z,\mathbf{Q},\boldsymbol{\tau})\|^2_{V(\mathcal{T},t)} &:= \|z\|^2 + t\|\nabla z\|^2_{\mathcal{T}} + \|\mathbf{Q}\|^2 + t\|\boldsymbol{\tau}\|^2 \\
&\quad + \|\nabla^s(\nabla z - t^2\mathbf{div}\,\mathbf{Q})\|^2_{\mathcal{T}} + \|\,\mathrm{div}(\mathbf{div}\,\mathbf{Q}+t(\boldsymbol{\tau}-\nabla z))\|^2_{\mathcal{T}}
\end{aligned}
$$

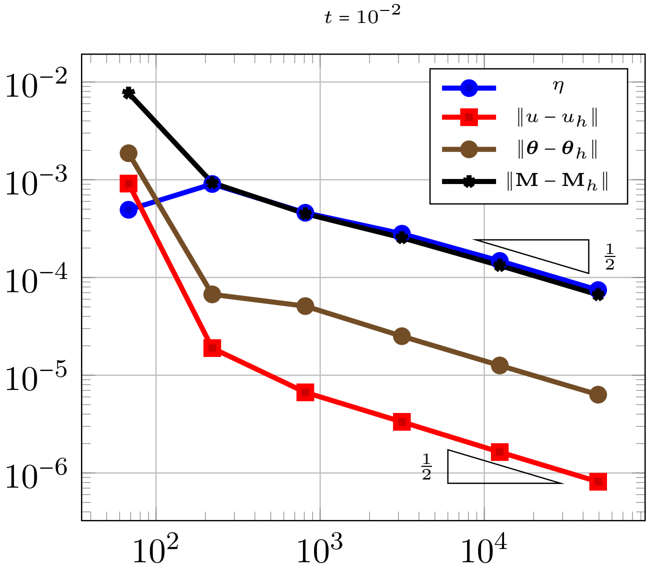
Reissner–Mindlin, Part 1

Main result

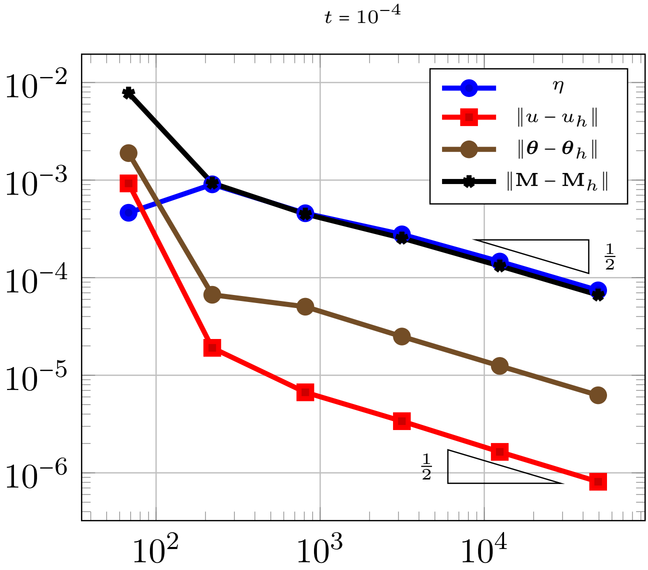
Theorem (Führer, Heuer, Sayas 2020).
The ultraweak variational formulation is uniformly well posed and the DPG approximation converges quasi-optimally.

What about the approximation of traces and shear locking?
Open. Smooth solutions are fine.

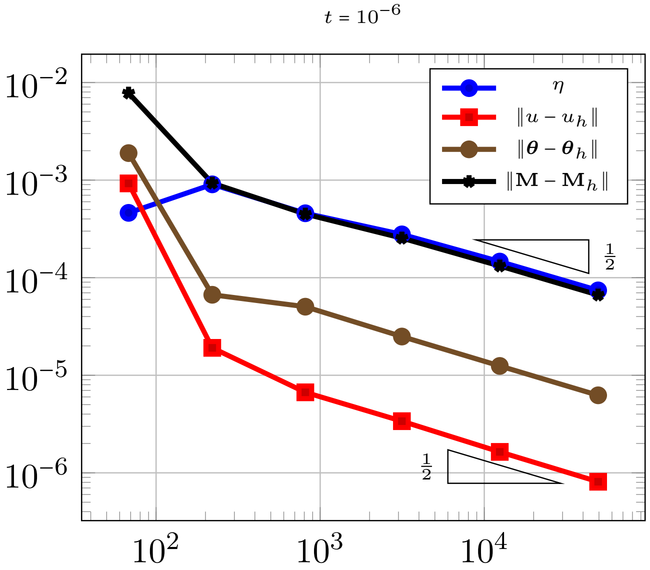
Polynomial solution



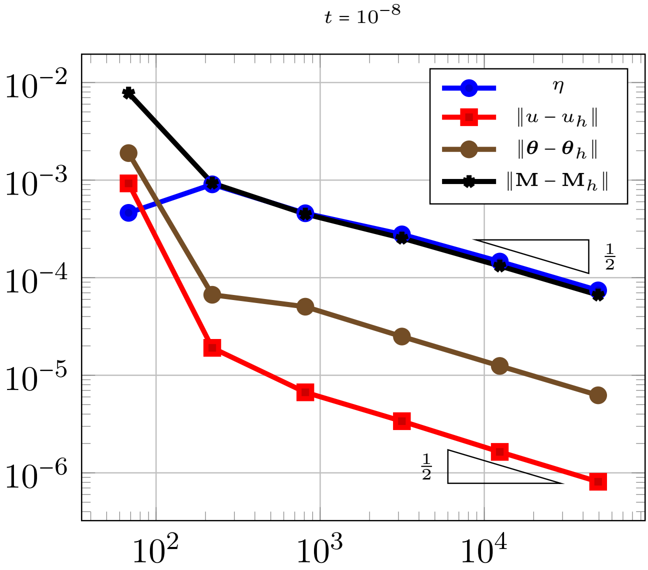
Polynomial solution



Polynomial solution



Polynomial solution



Timoshenko beam

Scaled Reissner–Mindlin:

$$-\operatorname{div} \mathbf{div}\, \mathbf{M} = f, \quad \mathbf{M} + \nabla^s(\nabla u - t^2 \mathbf{div}\, \mathbf{M}) = 0 \quad \text{in} \quad \Omega$$

Scaled Timoshenko beam:

$$-M'' = f, \quad M + u'' - t^2 M'' = 0 \quad \text{in} \quad I := (0, 1)$$

plus boundary conditions (for u, $\psi = u' - t^2 M'$, M, M').

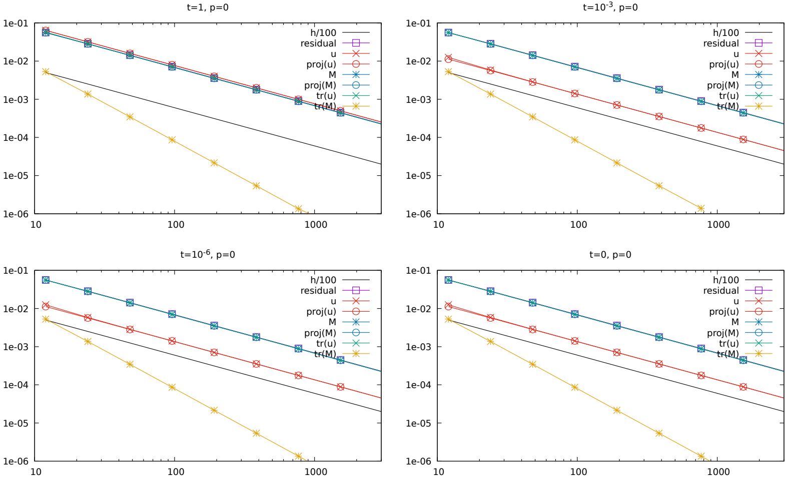
Full regularity: u, $M \in H^2(I)$.

We use standard $H^2(\mathcal{T})$-test norms and ultraweak formulation.

Theorem (Führer, García, Heuer 2021).
The ultraweak variational formulation is uniformly well posed and the DPG approximation converges quasi-optimally.

Timoshenko beam

$$f(x) = \sin(\pi x), \quad \text{clamped}(0), \text{free}(1), \ t = 1, 10^{-3}, 10^{-6}, 0.$$



Conclusions[1]

- A uniformly well-posed formulation does not guarantee locking-free approximations.
- For 1d problems, ultraweak formulations give locking-free schemes.
- The DPG framework is designed towards inf-sup stability,
  not to alleviate locking phenomena.
- Considering ultraweak formulations, traces carry all the burden:
  - regularity
  - conformity
  - locking phenomena
- The good news is: traces are lower-dimensional objects.

[1] Be patient, talk continues.

Some references

DPG: Niemi, Bramwell, Demkowicz ('11), Calo, Collier, Niemi ('14)
This talk so far: Führer, H., Sayas ('20), Führer, García, H. ('21)

DPG for Lamé: Bramwell, Demkowicz, Gopalakrishnan, Qiu ('12), Carstensen, Hellwig ('16)

Helmholtz decomposition: Brezzi, Fortin ('86), Arnold, Falk ('89)

Reduced integration: Brezzi, Bathe, Fortin ('89), Brezzi, Fortin, Stenberg ('91), Arnold, Falk ('97)

Least squares: Bramble, Sun ('98), Cai, Ye, Zhang ('99), Cai ('00)

Special elements: Arnold, Brezzi, Falk, Marini ('07)

Weakly over-penalized DG: Bösing, Carstensen ('15, '15)

Discrete de Rham (DDR) complex method: Di Pietro, Droniou (arXiv)

Reissner–Mindlin (scaled, $\mathcal{C} = \mathbf{I}$)

$$-\operatorname{div} \mathbf{div}\, \mathbf{M} = f, \qquad \mathbf{M} + \nabla^s(\nabla u - t^2 \mathbf{div}\, \mathbf{M}) = 0$$

with rotation and shear force variables:

$$-\operatorname{div} \boldsymbol{q} = f, \qquad \mathbf{M} + \nabla^s \boldsymbol{\psi} = 0, \qquad \mathbf{div}\, \mathbf{M} - \boldsymbol{q} = 0, \qquad \boldsymbol{q} = t^{-2}(\nabla u - \boldsymbol{\psi})$$

boundary conditions

$$\begin{aligned}
\text{hard clamped:} \quad & \boldsymbol{\psi} = 0,\ u = 0 && \text{on } \Gamma_{\mathrm{hc}}, \\
\text{soft clamped:} \quad & \boldsymbol{\psi} \cdot \boldsymbol{n} = \mathbf{t} \cdot \mathbf{M}\boldsymbol{n} = u = 0 && \text{on } \Gamma_{\mathrm{sc}}, \\
\text{hard simple support:} \quad & \boldsymbol{n} \cdot \mathbf{M}\boldsymbol{n} = \boldsymbol{\psi} \cdot \mathbf{t} = u = 0 && \text{on } \Gamma_{\mathrm{hss}}, \\
\text{soft simple support:} \quad & \mathbf{M}\boldsymbol{n} = 0,\ u = 0 && \text{on } \Gamma_{\mathrm{sss}}, \\
\text{free:} \quad & \mathbf{M}\boldsymbol{n} = 0,\ \boldsymbol{q} \cdot \boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{f}}.
\end{aligned}$$

Hemholtz decomposition

$$-\operatorname{div} \boldsymbol{q} = f, \qquad \mathbf{M} + \nabla^s \boldsymbol{\psi} = 0, \qquad \mathbf{div}\, \mathbf{M} - \boldsymbol{q} = 0, \qquad \boldsymbol{q} = t^{-2}(\nabla u - \boldsymbol{\psi})$$

$$\boxed{\boldsymbol{q} = \nabla r + \mathbf{curl}\, p}$$

with

$$r \in H^1_u(\Omega): \quad (\nabla r, \nabla \delta r) = (f, \delta r) \quad \forall \delta r \in H^1_u(\Omega)$$

$$p \in H^1(\Omega): \quad \operatorname{rot}\left(t^2\, \mathbf{curl}\, p + \boldsymbol{\psi}\right) = 0$$

$$\text{BC} \qquad p = 0 \quad \text{on } \Gamma_{\mathrm{f}}, \qquad \mathbf{t} \cdot (t^2\, \mathbf{curl}\, p + \boldsymbol{\psi}) = 0 \quad \text{on } \Gamma \setminus \bar{\Gamma}_{\mathrm{f}}$$

gives

$$\mathbf{M} + \nabla^s \boldsymbol{\psi} = 0, \quad \mathbf{div}\, \mathbf{M} - \mathbf{curl}\, p = \nabla r, \quad \operatorname{rot}(t\boldsymbol{\eta} + \boldsymbol{\psi}) = 0, \quad t\, \mathbf{curl}\, p - \boldsymbol{\eta} = 0$$

and u is determined by

$$u \in H^1_u(\Omega): \quad (\nabla u, \nabla \delta u) = t^2 (f, \delta u) + (\boldsymbol{\psi}, \nabla \delta u) \quad \forall \delta u \in H^1_u(\Omega).$$

Spaces and traces

Second stage:

$$\mathbf{M} + \nabla^s \psi = 0, \quad \mathbf{div}\,\mathbf{M} - \mathbf{curl}\,p = \nabla r, \quad \mathrm{rot}(t\eta + \psi) = 0, \quad t\,\mathbf{curl}\,p - \eta = 0$$

$$V_1(\mathcal{T}) := \boldsymbol{H}^1(\mathcal{T}) \times \boldsymbol{H}(\mathrm{rot}, \mathcal{T}), \qquad U_1 := \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}(\mathrm{rot}, \Omega) \; + \mathrm{BC}$$

$$\|(\psi, \eta)\|^2_{V_1(\mathcal{T},t)} := \|\psi\|^2 + \|\nabla\psi\|^2_{\mathcal{T}} + \|\eta\|^2 + t^{-2}\|\mathrm{rot}(t\eta + \psi)\|^2_{\mathcal{T}}$$

$$V_2(\mathcal{T}) := \mathbb{H}^s(\mathbf{div}, \mathcal{T}) \times H^1_*(\mathcal{T}), \qquad U_2 := \mathbb{H}(\mathbf{div}, \Omega) \times H^1(\Omega) \; + \mathrm{BC}$$

$$\|(\mathbf{M}, p)\|^2_{V_2(\mathcal{T},t)} := \|\mathbf{M}\|^2 + t_*^2\|p\|^2_* + \|\mathbf{div}\,\mathbf{M} - \mathbf{curl}\,p\|^2_{\mathcal{T}} + t^2\|\,\mathbf{curl}\,p\|^2_{\mathcal{T}}$$

$$\mathrm{tr}^{\psi} : \; U_1(\mathcal{T}) \to V_2(\mathcal{T})', \quad \langle \mathrm{tr}^{\psi}(\psi, \eta), (\mathbf{S}, v)\rangle_{\mathcal{S}} = \langle \psi, \mathbf{S}\boldsymbol{n} + v\mathbf{t}\rangle_{\mathcal{S}} + t\langle \eta \cdot \mathbf{t}, v\rangle_{\mathcal{S}}$$

$$\mathrm{tr}^{M} : \; U_2(\mathcal{T}) \to V_1(\mathcal{T})', \quad \langle \mathrm{tr}^{M}(\mathbf{M}, p), (\chi, \rho)\rangle_{\mathcal{S}} = \langle \mathbf{M}\boldsymbol{n}, \chi\rangle_{\mathcal{S}} + \langle p, (t\rho + \chi)\cdot\mathbf{t}\rangle_{\mathcal{S}}$$

Ultraweak formulation of second stage

$$U(t) := \mathbf{L}_2(\Omega) \times \mathbf{L}_2(\Omega) \times \mathbb{L}_2^s(\Omega) \times L_2^*(\Omega) \times \mathrm{tr}^{\psi}(U_1) \times \mathrm{tr}^{M}(U_2)$$

with norm^2 $\quad \|\boldsymbol{\psi}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\mathbf{M}\|^2 + t^2\|p\|_*^2 + \|\widehat{\psi\eta}\|_{\psi,t}^2 + \|\widehat{Mp}\|_{M,t}^2$

Find $(\boldsymbol{\psi}, \boldsymbol{\eta}, \mathbf{M}, p, \widehat{\psi\eta}, \widehat{Mp}) \in U(t)$*:*

$$(\boldsymbol{\psi}\,, \mathbf{curl}\, v - \mathbf{div}\,\mathbf{S})_{\mathcal{T}} + (\mathbf{M}\,, \mathbf{S} + \nabla^s\boldsymbol{\chi})_{\mathcal{T}} + (\boldsymbol{\eta}\,, t\,\mathbf{curl}\, v - \boldsymbol{\rho})_{\mathcal{T}}$$

$$+ (p\,, \mathrm{rot}(t\boldsymbol{\rho} + \boldsymbol{\chi}))_{\mathcal{T}} + \langle\widehat{\psi\eta}\,, (\mathbf{S}, v)\rangle_{\mathcal{S}} - \langle\widehat{Mp}\,, (\boldsymbol{\chi}, \boldsymbol{\rho})\rangle_{\mathcal{S}} = -(\nabla r\,, \boldsymbol{\chi}) \qquad (\star)$$

$$\forall (\boldsymbol{\chi}, \boldsymbol{\rho}, \mathbf{S}, v) \in V_1(\mathcal{T}) \times V_2(\mathcal{T}).$$

Theorem (Führer, Heuer, Niemi)
Given $t \in (0, 1]$, $r \in H_u^1(\Omega)$, $(\star)$ is well posed with solution

$$\|\boldsymbol{\psi}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\mathbf{M}\|^2 + t^2\|p\|_*^2 + \|\widehat{\psi\eta}\|_{\psi,t}^2 + \|\widehat{Mp}\|_{M,t}^2 \lesssim \|\nabla r\|^2.$$

Three-stage method

$$\text{FEM}\quad r_h \in X_h: \quad (\nabla r_h, \nabla \delta r) = (f, \delta r) \quad \forall \delta r \in X_h \subset H^1_u(\Omega)$$

$$\text{DPG}\quad \mathfrak{u}_h = (\psi_h, \eta_h, \mathbf{M}_h, p_h, \widehat{\psi\eta}_h, \widehat{Mp}_h) \in U_h:$$

$$b(\mathfrak{u}_h; \chi, \rho, \mathbf{S}, v) = -(\nabla r_h, \chi) \quad \forall (\chi, \rho, \mathbf{S}, v) \in \mathrm{T}(U_h)$$

$$\text{FEM}\quad u_h \in X_h: \quad (\nabla u_h, \nabla \delta u) = t^2 (f, \delta u) + (\psi_h, \nabla \delta u) \quad \forall \delta u \in X_h.$$

Cranking the DPG machinery ...

Theorem (Führer, Heuer, Niemi)
Given $t \in (0, 1]$, $f \in L_2(\Omega)$, FEM/DPG/FEM is well posed and

$$\|\nabla(r - r_h)\| + \|\mathfrak{u} - \mathfrak{u}_h\|_{U(t)} + \|\nabla(u - u_h)\| \lesssim \inf \|\nabla(r - \tilde{r}_h)\| + \|\mathfrak{u} - \tilde{\mathfrak{u}}_h\|_{U(t)} + \|\nabla(u - \tilde{u}_h)\|.$$

What about locking?

Three-stage method is locking free

Proposition (Arnold, Falk '89)
Conditions: hard-clamped, convex plate, $\mathcal{C} = \mathbf{I}$. Then

$$\|r\|_2 + \|u\|_2 + \|\psi\|_2 + \|p\|_1 + t\|p\|_2 \lesssim \|f\|$$

with $(p, 1) = 0$.

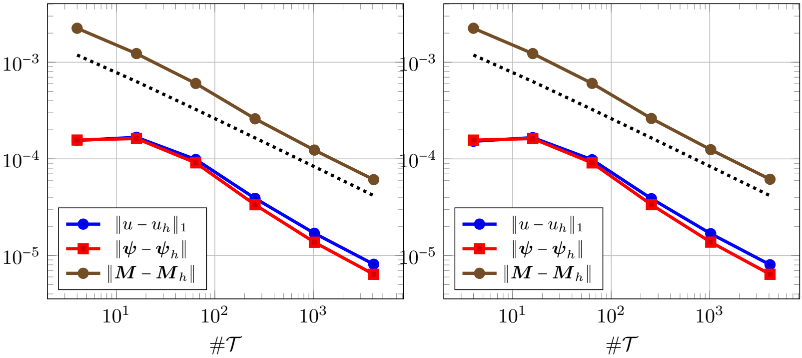
Theorem (Führer, Heuer, Niemi)
Under the conditions of the proposition, and for the lowest-order three-stage scheme,

$$\|\nabla(u - u_h)\| + \|\psi - \psi_h\| + \|\mathbf{M} - \mathbf{M}_h\| \lesssim h\|f\|$$

uniformly in $t \in (0, 1]$, $f \in L_2(\Omega)$, and $\mathcal{T}$.

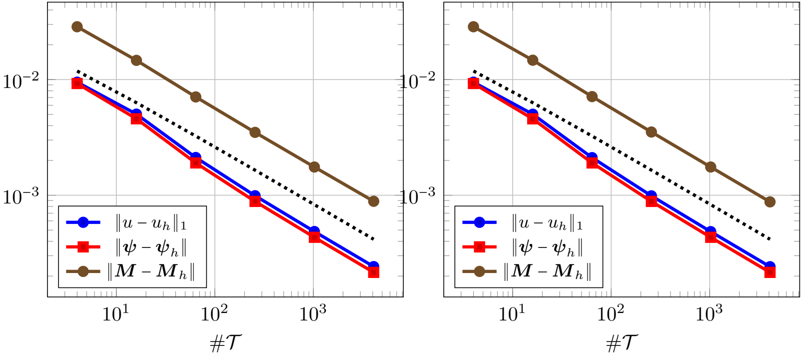
Polynomial solution

$\Omega = (0, 1)^2$, hc, $t = 10^{-2}, 10^{-4}$(left, right)



Kirchhoff solution (Fourier)

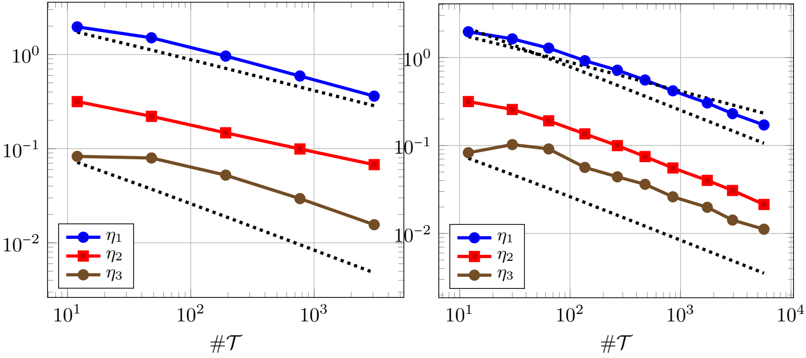
$$\Delta^2 u = 1, \quad \Omega = (0,1)^2, \quad \text{hss}, \quad t = 10^{-2}, 10^{-4} \text{(left, right)}$$



Singular solution (unknown)

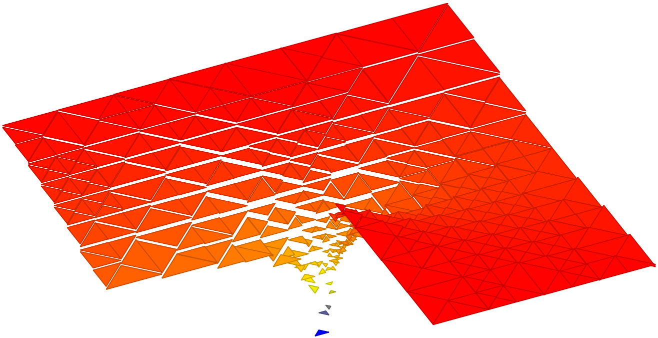
$$f = 1, \quad \Omega = (-1,1)^2 \setminus [-1,0], \quad \text{hc/f}, \quad t = 10^{-3}$$

uniform — adaptive



Singular solution (unknown)

$\mathbf{M}_{22}$ on adaptively refined mesh (477 elements)

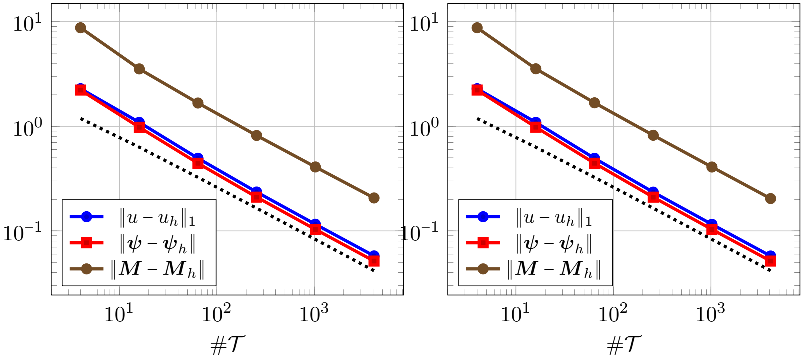


t-dependent solution

Manufactured solution from Di Pietro, Droniou (arXiv, '21)

$$\Omega = (0,1)^2, \quad \text{hc (non-homog.)}, \quad \|\boldsymbol{q}\|_1 \sim t^{-1/2}, \quad \operatorname{div} \boldsymbol{q} = -f = O(1)$$

$t = 10^{-2}$ $\qquad$ $t = 10^{-4}$



Final conclusions

Locking phenomena remain a tricky business.

Why use DPG/ultraweak formulations (in mechanics)?

- inf-sup stability
- flexibility in chosing variables ($\sim$ formulation)
- built-in adaptivity and error estimation
- symmetry of field tensor-variables
- convenience of traces (conformity, access to effective shear force, alleviate locking, . . .)