Towards adaptive hybrid high-order methods (HHO)

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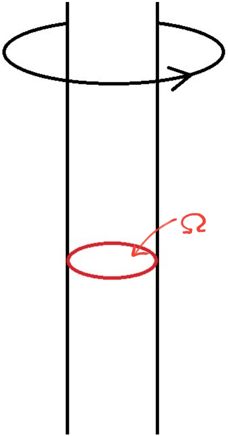
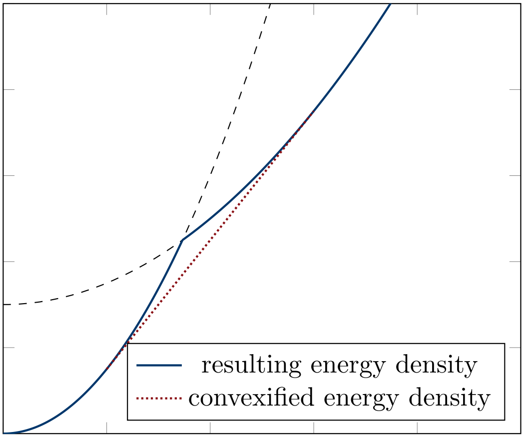
Optimal design problem





Kohn and Strang: Optimal design and relaxation of variational problems. I. Comm. Pure Appl. Math. (1986)

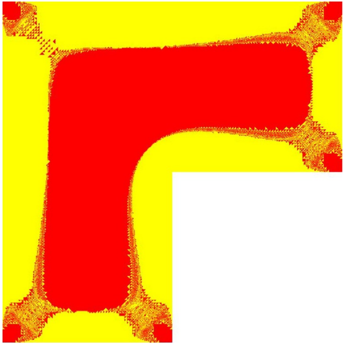
Bartels and Carstensen: A convergent adaptive finite element method for an optimal design problem. Numer. Math. (2008)

Optimal design problem

Parameters $0 < \mu_1 < \mu_2$ and $0 < t_1 < t_2$, $\mu_2 t_1 = \mu_1 t_2$ define energy density $W(F) := \psi(|F|)$ with

$$\psi(t) := \begin{cases} \mu_2 t^2/2 \text{ if } 0 \le t \le t_1, \\ t_1 \mu_2 (t - t_1/2) \text{ if } t_1 \le t \le t_2, \\ \mu_1 t^2/2 - t_1 \mu_2 (t_1/2 - t_2/2) \text{ if } t_2 \le t, \end{cases}$$

$$\psi'(t) := \begin{cases} \mu_2 t \text{ if } 0 \le t \le t_1, \\ \mu_2 t_1 \text{ if } t_1 \le t \le t_2, \\ \mu_1 t \text{ if } t_2 \le t \end{cases}$$



Mathematical setting

Given $\Omega \subset \mathbb{R}^n$ bdd. polyhedral Lipschitz, $2 \le p < \infty$, $1/p + 1/p' = 1$, suppose energy density $W \in C^1(\mathbb{M})$ satisfies, for all $A, B \in \mathbb{M} := \mathbb{R}^{m\times n}$, that

$$c_1|A|^p - c_2 \le W(A) \le c_3|A|^p + c_4, \tag{p-growth}$$

$$\begin{aligned}|\mathrm{D}W(A) - \mathrm{D}W(B)|^2 \le c_5(1 + |A|^{p-2} + |B|^{p-2})\\ \times (W(A) - W(B) - \mathrm{D}W(B) : (A - B))\end{aligned} \tag{cc}$$

Given $f \in L^{p'}(\Omega; \mathbb{R}^m)$, let u minimize

$$E(v) := \int_\Omega (W(\mathrm{D}v) - f \cdot v)\,\mathrm{d}x \quad \text{amongst } v \in V := W_0^{1,p}(\Omega; \mathbb{R}^m)$$

Dual problem

$W^*(G) := \sup_{A\in\mathbb{M}}(A : G - W(A))\ \forall G \in \mathbb{M}$. Given $G, H \in \mathbb{M}$, $A \in \partial W^*(G)$, $B \in \partial W^*(H)$

$$|G - H|^2 \le c_5(1 + |A|^{p-2} + |B|^{p-2}) \times (W^*(H) - W^*(G) - A : (H - G))$$

$\sigma := \mathrm{D}W(\mathrm{D}u)$ is unique with

$$\sigma \in \mathcal{Q}(f) := \{\tau \in \Sigma := W^{p'}(\mathrm{div}, \Omega; \mathbb{M}) : \mathrm{div}\,\tau + f = 0 \text{ in } \Omega\}$$

and maximizes

$$E^*(\tau) := -\int_\Omega W^*(\tau)\,\mathrm{d}x \quad \text{amongst } \tau \in \mathcal{Q}(f)$$

without duality gap $E(u) = \min E(\mathcal{A}) = \max E^*(\mathcal{Q}(f)) = E^*(\sigma)$

Dacorogna: Direct methods in the calculus of variations. Springer (2008)

Literature

C and Plecháč: Numerical solution of the scalar double-well problem allowing microstructure. Math. Comp. (1997)

Bartels and C: A convergent adaptive finite element method for an optimal design problem. Numer. Math. (2008)

C and Dolzmann: Convergence of adaptive finite element methods for a nonconvex double-well minimization problem. Math. Comp. (2015)

C, Günther, and Rabus: Mixed finite element method for a degenerate convex variational problem from topology optimization. SINUM (2012)

C and Liu: Nonconforming FEMs for an optimal design problem. SINUM (2015)

C and Tran: Unstabilized hybrid high-order method for a class of degenerate convex minimization problems. SINUM (2021)

C and Tran: Convergent adaptive hybrid higher-order schemes for convex minimization. Numer. Math. (2022)

Unstabilized HHO method

Discrete ansatz space

$V(\mathcal{T}) := \mathrm{P}_k(\mathcal{T};\mathbb{R}^m) \times \mathrm{P}_k(\mathcal{F}(\Omega);\mathbb{R}^m)$ on **simplicial** triangulation $\mathcal{T}$ of Ω with sides $\mathcal{F}$,
$\mathrm{I}v := (\Pi_{\mathcal{T}}^k v, \Pi_{\mathcal{F}}^k v) \in V(\mathcal{T})$ for all $v \in V$

Gradient reconstruction

Let $\Sigma(\mathcal{T}) := \mathrm{RT}_k^{\mathrm{pw}}(\mathcal{T};\mathbb{M})$, $\mathcal{G}v_h \in \Sigma(\mathcal{T})$ of $v_h = (v_{\mathcal{T}}, v_{\mathcal{F}}) \in V(\mathcal{T})$ is unique solution to

$$\int_\Omega \mathcal{G}v_h : \tau_h \,\mathrm{d}x = -\int_\Omega v_{\mathcal{T}} \cdot \mathrm{div}_{\mathrm{pw}}\, \tau_h \,\mathrm{d}x + \sum_{F\in\mathcal{F}} \int_F \underbrace{v_F}_{=(v_{\mathcal{F}})|_F} \cdot [\tau_h \nu_F]_F \,\mathrm{d}s \quad \text{for all } \tau_h \in \Sigma(\mathcal{T})$$

- $\|\mathcal{G}\bullet\|_{L^p(\Omega)}$ is a norm in $V(\mathcal{T}) \to$ no stabilization
- $\Pi_{\Sigma(\mathcal{T})}\mathrm{D}v = \mathcal{G}\mathrm{I}v$

Abbas, Ern, and Pignet: Hybrid high-order methods for finite deformations of hyperelastic materials. Comput. Mech. (2018)

Discrete minimization problem

Let $u_h \in V(\mathcal{T})$ minimize

$$E_h(v_h) := \int_\Omega (W(\mathcal{G} v_h) - f \cdot v_\mathcal{T})\,\mathrm{d}x \quad \text{amongst } v_h = (v_\mathcal{T}, v_\mathcal{F}) \in V(\mathcal{T})$$

C and Liu: Nonconforming FEMs for an optimal design problem. SINUM (2015)

Discrete minimization problem

Let $u_h \in V(\mathcal{T})$ minimize

$$E_h(v_h) := \int_\Omega \big(W(\mathcal{G}v_h) - f \cdot v_\mathcal{T}\big)\,\mathrm{d}x \quad \text{amongst } v_h = (v_\mathcal{T}, v_\mathcal{F}) \in V(\mathcal{T})$$

Lemma (σ_h is $H(\mathrm{div})$ conform)

$\mathrm{D}W(\mathcal{G}u_h)$ and $\sigma_h := \Pi_{\Sigma(\mathcal{T})}\mathrm{D}W(\mathcal{G}u_h)$ are unique and

$$\sigma_h \in \mathcal{Q}(f,\mathcal{T}) := \{\tau_h \in \mathrm{RT}_k(\mathcal{T};\mathbb{M}) : \mathrm{div}\,\tau_h + \Pi_\mathcal{T}^k f = 0 \text{ in } \Omega\}$$

C and Liu: Nonconforming FEMs for an optimal design problem. SINUM (2015)

$H(\mathrm{div})$ conformity

Proof.

$$\int_\Omega \overset{\sigma_h}{\cancel{\mathrm{D}W(\mathcal{G}u_h)}} : \mathcal{G}v_h \,\mathrm{d}x = \int_\Omega f \cdot v_{\mathcal{T}} \,\mathrm{d}x \quad \forall v_h \in V(\mathcal{T}) \tag{dELE}$$

$H(\mathrm{div})$ conformity

Proof.

$$\int_\Omega \cancel{\mathrm{D}W(\overset{\sigma_h}{\mathcal{G}u_h})} : \mathcal{G}v_h \, \mathrm{d}x = \int_\Omega f \cdot v_{\mathcal{T}} \, \mathrm{d}x \quad \forall v_h \in V(\mathcal{T}) \tag{dELE}$$

Fix $F \in \mathcal{F}(\Omega)$, $v_h = (0, v_{\mathcal{F}}) \in V(\mathcal{T})$ with $(v_{\mathcal{F}})|_E = 0 \ \forall E \in \mathcal{F} \setminus \{F\}$ in (dELE) $\implies$

$$0 = \int_\Omega \sigma_h : \mathcal{G}v_h \, \mathrm{d}x = \cancel{- \int_\Omega v_{\mathcal{T}} \cdot \mathrm{div}_{\mathrm{pw}} \, \sigma_h \, \mathrm{d}x} + \int_F v_F \cdot [\sigma_h \nu_F]_F \, \mathrm{d}s$$

$H(\mathrm{div})$ conformity

Proof.

$$\int_\Omega \overset{\sigma_h}{\cancel{\mathrm{D}W(\mathcal{G}u_h)}} : \mathcal{G}v_h \,\mathrm{d}x = \int_\Omega f \cdot v_{\mathcal{T}} \,\mathrm{d}x \quad \forall v_h \in V(\mathcal{T}) \tag{dELE}$$

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$$0 = \int_\Omega \sigma_h : \mathcal{G}v_h \,\mathrm{d}x = \cancel{-\int_\Omega v_{\mathcal{T}} \cdot \mathrm{div}_{\mathrm{pw}}\, \sigma_h \,\mathrm{d}x} + \int_F v_F \cdot [\sigma_h \nu_F]_F \,\mathrm{d}s$$

$\implies [\sigma_h \nu_F]_F \perp P_k(F; \mathbb{R}^m) \implies [\sigma_h \nu_F]_F = 0$ on $F \in \mathcal{F}(\Omega) \implies \sigma_h \in \mathrm{RT}_k(\mathcal{T}; \mathbb{M})$

$H(\mathrm{div})$ conformity

Proof.

$$\int_\Omega \cancel{\mathrm{D}W(\mathcal{G}u_h)}^{\sigma_h} : \mathcal{G}v_h \, \mathrm{d}x = \int_\Omega f \cdot v_\mathcal{T} \, \mathrm{d}x \quad \forall v_h \in V(\mathcal{T}) \tag{dELE}$$

Fix $F \in \mathcal{F}(\Omega)$, $v_h = (0, v_\mathcal{F}) \in V(\mathcal{T})$ with $(v_\mathcal{F})|_E = 0 \ \forall E \in \mathcal{F} \setminus \{F\}$ in (dELE) $\implies$

$$0 = \int_\Omega \sigma_h : \mathcal{G}v_h \, \mathrm{d}x = -\cancel{\int_\Omega v_\mathcal{T} \cdot \mathrm{div}_{\mathrm{pw}} \sigma_h \, \mathrm{d}x} + \int_F v_F \cdot [\sigma_h \nu_F]_F \, \mathrm{d}s$$

$\implies [\sigma_h \nu_F]_F \perp P_k(F; \mathbb{R}^m) \implies [\sigma_h \nu_F]_F = 0$ on $F \in \mathcal{F}(\Omega) \implies \sigma_h \in \mathrm{RT}_k(\mathcal{T}; \mathbb{M})$

$v_h = (v_\mathcal{T}, 0) \in V(\mathcal{T})$ in (dELE) $\implies \int_\Omega f \cdot v_\mathcal{T} \, \mathrm{d}x = -\int_\Omega v_\mathcal{T} \cdot \mathrm{div}\, \sigma_h \, \mathrm{d}x \implies \mathrm{div}\, \sigma_h + \Pi_\mathcal{T}^k f = 0$

Convexity control

Let $\alpha := (2-p')/p'$. For any $\xi, \varrho \in L^p(\Omega;\mathbb{M})$,

$$\|\mathrm{D}W(\xi) - \mathrm{D}W(\varrho)\|^2_{L^{p'}(\Omega)} \le 3c_5(|\Omega| + \|\xi\|^p_{L^p(\Omega)} + \|\varrho\|^p_{L^p(\Omega)})^\alpha \times \int_\Omega (W(\xi) - W(\varrho) - \mathrm{D}W(\varrho) : (\xi - \varrho))\,\mathrm{d}x \tag{cc}$$

For any $\tau, \phi \in L^{p'}(\Omega;\mathbb{M})$ and $\xi, \varrho \in L^p(\Omega;\mathbb{M})$ s.t. $\xi \in \partial W^*(\tau)$ and $\varrho \in \partial W^*(\phi)$ a.e. in Ω,

$$\|\tau - \phi\|^2_{L^{p'}(\Omega)} \le 3c_5(|\Omega| + \|\xi\|^p_{L^p(\Omega)} + \|\varrho\|^p_{L^p(\Omega)})^\alpha \times \int_\Omega (W^*(\tau) - W^*(\phi) - \varrho : (\tau - \phi))\,\mathrm{d}x \tag{dual-cc}$$

C and Plecháč: Numerical solution of the scalar double-well problem allowing microstructure. Math. Comp. (1997)
C and T: Unstabilized hybrid high-order method for a class of degenerate convex minimization problems. SINUM (2021)

A priori error analysis

Theorem (a priori)

$$\|\sigma - \sigma_h\|^2_{L^{p'}(\Omega)} + \|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|^2_{L^{p'}(\Omega)} + |E(u) - E_h(u_h)|$$
$$\lesssim |E^*(\sigma) - \max E^*(\mathcal{Q}(f,\mathcal{T}))| + \mathrm{osc}(f,\mathcal{T}) + \|(1 - \Pi_{\Sigma(\mathcal{T})})\mathrm{D}u\|^2_{L^p(\Omega)}$$

Corollary (convergence rates)

If $\sigma \in W^{k+1,p'}(\mathcal{T};\mathbb{M}) \cap W^{1,p'}(\Omega;\mathbb{M})$, $u \in W^{k+2}(\mathcal{T};\mathbb{R}^m) \cap V$, then

$$\|\sigma - \sigma_h\|^2_{L^{p'}(\Omega)} + \|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|^2_{L^{p'}(\Omega)} + |E(u) - E_h(u_h)| \lesssim h_{\max}^{k+1}$$

Lower energy bound

For $C_1 \geq 3c_5(|\Omega| + C_2^p + \|\partial W^*(\sigma_h)\|^p_{L^p(\Omega)})^\alpha$ with $\|\mathrm{D}u\|_{L^p(\Omega)} \leq C_2$, (dual-cc) $\implies$

$$C_1^{-1}\|\sigma - \sigma_h\|^2_{L^{p'}(\Omega)} \leq \int_\Omega \Big(W^*(\sigma_h) - W^*(\sigma) \underbrace{-\mathrm{D}u : (\sigma - \sigma_h)}_{=-\int_\Omega u\cdot(1-\Pi_\mathcal{T}^k)f\,\mathrm{d}x \leq C_2 \operatorname{osc}(f,\mathcal{T})}\Big)\,\mathrm{d}x$$

Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

Lower energy bound

For $C_1 \geq 3c_5(|\Omega| + C_2^p + \|\partial W^*(\sigma_h)\|^p_{L^p(\Omega)})^\alpha$ with $\|\mathrm{D}u\|_{L^p(\Omega)} \leq C_2$, (dual-cc) $\Longrightarrow$

$$C_1^{-1}\|\sigma - \sigma_h\|^2_{L^{p'}(\Omega)} \leq \int_\Omega (W^*(\sigma_h) - W^*(\sigma) \underbrace{-\mathrm{D}u : (\sigma - \sigma_h)}_{=-\int_\Omega u\cdot(1-\Pi_\mathcal{T}^k)f\,\mathrm{d}x \leq C_2 \operatorname{osc}(f,\mathcal{T})})\,\mathrm{d}x$$

Lemma (LEB)

$$C_1^{-1}\|\sigma - \sigma_h\|^2_{L^{p'}(\Omega)} \leq E^*(\sigma) - E^*(\sigma_h) + C_2 \operatorname{osc}(f,\mathcal{T})$$

- superior to $E_h(u_h) - C\|h_\mathcal{T} f\|_{L^{p'}(\Omega)} \leq E(u)$ for CR-FEM

Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

A posteriori error estimate (cont.)

For $C_3 \geq 3c_5(|\Omega| + C_2^p + \|\mathcal{G}u_h\|^p_{L^p(\Omega)})^\alpha$, (cc-dual) $\implies$

$$
\begin{aligned}
&C_3^{-1}\|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|^2_{L^{p'}(\Omega)} \\
&\qquad \leq \int_\Omega (W^*(\sigma) - W^*(\mathrm{D}W(\mathcal{G}u_h)) - (\mathcal{G}u_h - \mathrm{D}v) : (\sigma - \sigma_h))\,\mathrm{d}x \underbrace{- \int_\Omega \mathrm{D}v : (\sigma - \sigma_h)\,\mathrm{d}x}_{-\int_\Omega (1-\Pi_{\mathcal{T}}^k) f\cdot v\,\mathrm{d}x}
\end{aligned}
$$

A posteriori error estimate (cont.)

For $C_3 \geq 3c_5(|\Omega| + C_2^p + \|\mathcal{G}u_h\|^p_{L^p(\Omega)})^\alpha$, (cc-dual) $\Longrightarrow$

$$C_3^{-1}\|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|^2_{L^{p'}(\Omega)}$$
$$\leq \int_\Omega (W^*(\sigma) - W^*(\mathrm{D}W(\mathcal{G}u_h)) - (\mathcal{G}u_h - \mathrm{D}v) : (\sigma - \sigma_h))\,\mathrm{d}x \underbrace{- \int_\Omega \mathrm{D}v : (\sigma - \sigma_h)\,\mathrm{d}x}_{-\int_\Omega (1-\Pi^k_{\mathcal{T}}) f \cdot v\,\mathrm{d}x}$$

NB: $-\int_\Omega W^*(\mathrm{D}W(\mathcal{G}u_h))\,\mathrm{d}x = \int_\Omega (W(\mathcal{G}u_h) - \sigma_h : \mathcal{G}u_h)\,\mathrm{d}x = E_h(u_h)$

A posteriori error estimate (cont.)

For $C_3 \geq 3c_5(|\Omega| + C_2^p + \|\mathcal{G}u_h\|_{L^p(\Omega)}^p)^\alpha$, (cc-dual) $\Longrightarrow$

$$C_3^{-1}\|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|_{L^{p'}(\Omega)}^2 \leq \int_\Omega (W^*(\sigma) - W^*(\mathrm{D}W(\mathcal{G}u_h)) - (\mathcal{G}u_h - \mathrm{D}v) : (\sigma - \sigma_h))\,\mathrm{d}x \underbrace{- \int_\Omega \mathrm{D}v : (\sigma - \sigma_h)\,\mathrm{d}x}_{-\int_\Omega (1-\Pi_\mathcal{T}^k) f\cdot v\,\mathrm{d}x}$$

NB: $$-\int_\Omega W^*(\mathrm{D}W(\mathcal{G}u_h))\,\mathrm{d}x = \int_\Omega (W(\mathcal{G}u_h) - \sigma_h : \mathcal{G}u_h)\,\mathrm{d}x = E_h(u_h)$$

$$\leq E_h(u_h) - E(u) - \|\sigma - \sigma_h\|_{L^{p'}(\Omega)}\|\mathcal{G}u_h - \mathrm{D}v\|_{L^p(\Omega)} - \int_\Omega (1 - \Pi_\mathcal{T}^k) f \cdot v\,\mathrm{d}x$$

Recall $C_1^{-1}\|\sigma - \sigma_h\|_{L^{p'}(\Omega)}^2 + E^*(\sigma_h) - C_2\,\mathrm{osc}(f, \mathcal{T}) \leq E(u)$

A posteriori error estimate

Theorem (a posteriori)

$$\|\sigma - \sigma_h\|^2_{L^{p'}(\Omega)} + \|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|^2_{L^{p'}(\Omega)} + |E(u) - E_h(u_h)|$$
$$\lesssim E_h(u_h) - E^*(\sigma_h) + \operatorname{osc}(f, \mathcal{T}) + \min_{v \in V} \|\mathcal{G}u_h - \mathrm{D}v\|^2_{L^p(\Omega)} =: \mathrm{RHS}$$

- $E_h(u_h) - E^*(\sigma_h) = 0$ iff $\mathrm{D}W(\mathcal{G}u_h) \in \Sigma(\mathcal{T})$
- $E_h(u_h) - E^*(\sigma_h) > 0$ to be expected in general $\to$ discrete duality gap
- RHS computable with post-processing of $\mathcal{G}u_h$ in V, e.g. by averaging or right inverse $\mathcal{J} : V(\mathcal{T}) \to V$

Adaptive algorithm

INPUT: $\mathcal{T}_0$, $0 < \varepsilon \leq k+1$, $0 < \theta < 1$, $k \geq 0$

$$\begin{tikzcd} \boxed{\text{Solve}} \arrow[r] & \boxed{\text{Estimate}} \arrow[r] & \boxed{\text{Mark}} \arrow[r] & \boxed{\text{Refine}} \arrow[lll, bend left] \end{tikzcd}$$

Refinement indicator:

$$\begin{aligned} \eta_\ell^{(\varepsilon)}(T) := |T|^{(\varepsilon p - p)/n} \|\Pi_T^k(\mathcal{R}_\ell u_\ell - u_T)\|_{L^p(T)}^p + |T|^{\varepsilon p'/n} \|\sigma_\ell - \mathrm{D}W(\mathcal{G} u_\ell)\|_{L^{p'}(T)}^{p'} \\ + |T|^{(\varepsilon p + 1 - p)/n} \Big(\sum_{F \in \mathcal{F}_\ell(T)} \|[\mathcal{R}_\ell u_\ell]_F\|_{L^p(F)}^p + \sum_{F \in \mathcal{F}_\ell(T)} \|\Pi_F^k((\mathcal{R}_\ell u_\ell)|_T - u_F)\|_{L^p(F)}^p \Big) \\ + |T|^{p'/n} \|(1 - \Pi_T^k) f\|_{L^{p'}(T)}^{p'} \end{aligned}$$

Output: $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$, $(u_\ell)_{\ell \in \mathbb{N}_0}$, $(\sigma_\ell)_{\ell \in \mathbb{N}_0}$

Plain convergence

Theorem (Plain convergence)

Let W satisfies (p-growth). Given $\mathcal{T}_0$, $0 < \varepsilon \leq k+1$, $0 < \theta < 1$, let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$, $(u_\ell)_{\ell\in\mathbb{N}_0}$, $(\sigma_\ell)_{\ell\in\mathbb{N}_0}$ be the output of the adaptive algorithm

(a) $\lim_{\ell\to\infty} E_\ell(u_\ell) = E(u)$

(b) If W satisfies (cc), then $\lim_{\ell\to\infty} \mathrm{D}W(\mathcal{G}_\ell u_\ell) = \sigma$ and $\sigma_\ell \rightharpoonup \sigma$ in $L^{p'}(\Omega;\mathbb{M})$

(c) If W is strongly convex, then $\lim_{\ell\to\infty} \mathcal{G}_\ell u_\ell = \mathrm{D}u$ in $L^p(\Omega;\mathbb{R}^m)$

Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

Balci, Ortner, Storn: Crouzeix-Raviart finite element method for non-autonomous variational problems with Lavrentiev gap. Numer. Math. (2022)

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Let W satisfies (p-growth). Given $\mathcal{T}_0$, $0 < \varepsilon \leq k+1$, $0 < \theta < 1$, let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$, $(u_\ell)_{\ell\in\mathbb{N}_0}$, $(\sigma_\ell)_{\ell\in\mathbb{N}_0}$ be the output of the adaptive algorithm

(a) $\lim_{\ell\to\infty} E_\ell(u_\ell) = E(u)$

(b) If W satisfies (cc), then $\lim_{\ell\to\infty} \mathrm{D}W(\mathcal{G}_\ell u_\ell) = \sigma$ and $\sigma_\ell \rightharpoonup \sigma$ in $L^{p'}(\Omega;\mathbb{M})$

(c) If W is strongly convex, then $\lim_{\ell\to\infty} \mathcal{G}_\ell u_\ell = \mathrm{D}u$ in $L^p(\Omega;\mathbb{R}^m)$

- Upper growth is required $\to$ ~~not applicable to Lavrentiev gap~~
- Applicable to Lavrentiev gap for $k = 0$ and for $k \geq 1$ under additional assumptions

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Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

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Outline of the proof

1. (Convergence of $\eta_\ell^{(\varepsilon)}$) Prove that $\lim_{\ell\to\infty} \eta_\ell^{(\varepsilon)} = 0$
2. (Discrete compactness) Prove that $\lim_{\ell\to\infty} \eta_\ell^{(\varepsilon)} = 0$ implies $\mathcal{J}_\ell u_\ell \rightharpoonup v$ in V, $\mathcal{G}_\ell u_\ell \rightharpoonup \mathrm{D}v$ in $L^p(\Omega;\mathbb{M})$, and $\sigma_\ell \rightharpoonup \mathrm{D}W(\mathcal{G}_\ell u_\ell)$ in $L^{p'}(\Omega;\mathbb{M})$
3. (LEB) Establish a (not computable) LEB $\mathrm{LEB}_\ell \leq E(u)$
4. (Convergence of $E_\ell(u_\ell)$) Apply discrete compactness to show $\lim_{\ell\to\infty} \mathrm{LEB}_\ell = E(v)$ and then $\lim_{\ell\to\infty} E_\ell(u_\ell) = \lim_{\ell\to\infty} \mathrm{LEB}_\ell = E(u)$
5. (Convergence of stress/displacement) Use (convexity) control over $\|\sigma - \mathrm{D}W(\mathcal{G}_\ell u_\ell)\|_{L^{p'}(\Omega)}$ or $\|\mathrm{D}u - \mathcal{G}_\ell u_\ell\|_{L^p(\Omega)}$

Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

Convergence of $\eta_\ell^{(\varepsilon)}$

$$\mu_\ell^{(0)}(T) := |T|^{-p/n}\|\Pi_T^k(\mathcal{R}_\ell u_\ell - u_T)\|_{L^p(T)}^p + |T|^{(1-p)/n}\Big(\sum_{F\in\mathcal{F}_\ell(T)}\|[\mathcal{R}_\ell u_\ell]_F\|_{L^p(F)}^p$$
$$+\sum_{F\in\mathcal{F}_\ell(T)}\|\Pi_F^k((\mathcal{R}_\ell u_\ell)|_T - u_F)\|_{L^p(F)}^p\Big) \lesssim \|u_\ell\|_{\ell,T}^p \approx \|\mathcal{G}_\ell u_\ell\|_{L^p(T)}^p$$

Nochetto and Veeser: Primer of Adaptive Finite Element Methods. Lecture Notes in Mathematics (2012)

Convergence of $\eta_\ell^{(\varepsilon)}$

$$\mu_\ell^{(0)}(T) := |T|^{-p/n} \|\Pi_T^k(\mathcal{R}_\ell u_\ell - u_T)\|_{L^p(T)}^p + |T|^{(1-p)/n} \Big(\sum_{F \in \mathcal{F}_\ell(T)} \|[\mathcal{R}_\ell u_\ell]_F\|_{L^p(F)}^p$$
$$+ \sum_{F \in \mathcal{F}_\ell(T)} \|\Pi_F^k((\mathcal{R}_\ell u_\ell)|_T - u_F)\|_{L^p(F)}^p \Big) \lesssim \|u_\ell\|_{\ell,T}^p \approx \|\mathcal{G}_\ell u_\ell\|_{L^p(T)}^p$$

Let $\Omega_\ell := \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$,

$$\eta_\ell^{(\varepsilon)}(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}) \lesssim \|h_\ell\|_{L^\infty(\Omega_\ell)}^{\varepsilon p'} \|\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)\|_{L^{p'}(\Omega)}^{p'} + \|h_\ell\|_{L^\infty(\Omega_\ell)}^{p'} \|f\|_{L^{p'}(\Omega)}$$
$$+ \|h_\ell\|_{L^\infty(\Omega_\ell)}^{\varepsilon p} \mu_\ell^{(0)}(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})$$

$\lim_{\ell \to \infty} \|h_\ell\|_{L^\infty(\Omega_\ell)} = 0 \implies \lim_{\ell \to \infty} \eta_\ell^{(\varepsilon)}(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}) = 0$

Dörfler marking $\implies \eta_\ell^{(\varepsilon)} \le \theta^{-1} \eta_\ell^{(\varepsilon)}(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}) \implies \lim_{\ell \to \infty} \eta_\ell^{(\varepsilon)} = 0$ $\square$

Nochetto and Veeser: Primer of Adaptive Finite Element Methods. Lecture Notes in Mathematics (2012)

Conforming companion

Lemma (right-inverse)

There exist linear bounded operator $\mathcal{J} : V(\mathcal{T}) \to V$ s.t., for all $v_h = (v_\mathcal{T}, v_\mathcal{F}) \in V(\mathcal{T})$,

$$\Pi_\mathcal{T}^k \mathcal{J} v_h = v_\mathcal{T}, \quad \Pi_\mathcal{F}^k \mathcal{J} v_h = v_\mathcal{F}, \quad \|\mathrm{D}\mathcal{J} v_h\|_{L^p(\Omega)} \lesssim \|\mathcal{G} v_h\|_{L^p(\Omega)}$$

and

$$\|\mathcal{G} v_h - \mathrm{D}\mathcal{J} v_h\|_{L^p(T)}^p \lesssim \mu_\ell^{(0)}(T)$$

- $\mathrm{I}\mathcal{J} = \mathrm{Id}$ in $V(\mathcal{T})$
- $\mathrm{D}\mathcal{J} v_h - \mathcal{G} v_h \perp \Sigma(\mathcal{T}) = \mathrm{RT}_k^{\mathrm{pw}}(\mathcal{T}; \mathbb{M})$

Verfürth: A Posteriori Error Estimation Techniques for Finite Element Methods. Oxford University Press (2013)

Ern and Zanotti: A quasi-optimal variant of the hybrid high-order method for elliptic partial differential equations with H^{-1} loads. IMA J. Numer. Anal. (2020)

C, Gallistl, Schedensack: Adaptive nonconforming Crouzeix-Raviart FEM for eigenvalue problems. Math. Comp. (2015)

Discrete compactness

Stability of $\mathcal{J}_\ell$ and discrete Sobolev embedding $\implies$

$$\|\mathrm{D}\mathcal{J}_\ell u_\ell\|_{L^p(\Omega)} \lesssim \|\mathcal{G}_\ell u_\ell\|_{L^p(\Omega)} \lesssim 1$$

Banach-Alaoglu theorem $\implies$ (not relabelled) subsequence of $(u_\ell)_{\ell\in\mathbb{N}_0}$, $v \in W^{1,p}(\Omega;\mathbb{R}^m)$, $G \in L^p(\Omega;\mathbb{M})$ s.t. $\mathcal{J}_\ell u_\ell \rightharpoonup v$ weakly in V and $\mathcal{G}_\ell u_\ell \rightharpoonup G$ weakly in $L^p(\Omega;\mathbb{M})$

$$\underbrace{\int_\Omega \mathcal{G}_\ell u_\ell : \Phi \,\mathrm{d}x}_{\to\int_\Omega G:\Phi\,\mathrm{d}x} = \int_\Omega (\mathcal{G}_\ell u_\ell - \mathrm{D}\mathcal{J}_\ell u_\ell) : \Phi \,\mathrm{d}x + \int_\Omega \mathrm{D}\mathcal{J}_\ell u_\ell : \Phi \,\mathrm{d}x$$

$$= \underbrace{\int_\Omega (\mathcal{G}_\ell u_\ell - \mathrm{D}\mathcal{J}_\ell u_\ell) : (1 - \Pi_{\Sigma(\mathcal{T}_\ell)})\Phi \,\mathrm{d}x}_{\lesssim \|h_\ell^{k+1}(\mathcal{G}_\ell u_\ell - \mathrm{D}\mathcal{J}_\ell u_\ell)\|_{L^p(\Omega)} |\Phi|_{W^{k+1,p'}(\Omega)}} - \underbrace{\int_\Omega \mathcal{J}_\ell u_\ell \cdot \operatorname{div}\varphi \,\mathrm{d}x}_{\to\int_\Omega v\cdot\operatorname{div}\Phi\,\mathrm{d}x}$$

Discrete compactness (cont.)

$$\|h_\ell^{k+1}(\mathcal{G}_\ell u_\ell - \mathrm{D}\mathcal{J}_\ell u_\ell)\|_{L^p(\Omega)} \lesssim \mu_\ell^{(k+1)} \le \eta_\ell^{(k+1)} \lesssim \eta_\ell^{(\varepsilon)} \to 0 \implies$$

$$\int_\Omega G : \Phi \,\mathrm{d}x = -\int_\Omega v \cdot \operatorname{div} \Phi \,\mathrm{d}x \;\forall \Phi \in C^\infty(\overline{\Omega}; \mathbb{M}) \implies G = \mathrm{D}v$$

In conclusion, $\mathcal{J}_\ell u_\ell \rightharpoonup v$ in V and $\mathcal{G}_\ell u_\ell \rightharpoonup \mathrm{D}v$ □

Discrete compactness (cont.)

$$\|h_\ell^{k+1}(\mathcal{G}_\ell u_\ell - \mathrm{D}\mathcal{J}_\ell u_\ell)\|_{L^p(\Omega)} \lesssim \mu_\ell^{(k+1)} \leq \eta_\ell^{(k+1)} \lesssim \eta_\ell^{(\varepsilon)} \to 0 \implies$$

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In conclusion, $\mathcal{J}_\ell u_\ell \rightharpoonup v$ in V and $\mathcal{G}_\ell u_\ell \rightharpoonup \mathrm{D}v$ $\square$

For all $\Phi \in C^\infty(\Omega; \mathbb{M})$,

$$\left|\int_\Omega (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : \Phi \,\mathrm{d}x\right| = \left|\int_\Omega (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : (1 - \Pi_{\Sigma(\mathcal{T}_\ell)})\Phi \,\mathrm{d}x\right|$$

$$\lesssim \|h_\ell^{k+1}(\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell))\|_{L^{p'}(\Omega)} |\Phi|_{W^{k+1,p}(\Omega)} \to 0$$

$$\implies \sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell) \rightharpoonup 0 \text{ in } L^{p'}(\Omega; \mathbb{M})$$ $\square$

LEB

$$
\begin{aligned}
0 &\leq \int_\Omega \left(W(\mathrm{D}u) - W(\mathcal{G}_\ell u_\ell) - \mathrm{D}W(\mathcal{G}_\ell u_\ell) : (\mathrm{D}u - \mathcal{G}_\ell u_\ell)\right) \mathrm{d}x \\
&= \int_\Omega \left(W(\mathrm{D}u) - W(\mathcal{G}_\ell u_\ell) + (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : (\mathrm{D}u - \mathcal{G}_\ell u_\ell)\right) \mathrm{d}x - \underbrace{\int_\Omega \sigma_\ell : (\mathrm{D}u - \mathcal{G}_\ell u_\ell)\,\mathrm{d}x}_{\int_\Omega \Pi^k_{\mathcal{T}_\ell} f \cdot (u - u_{\mathcal{T}_\ell})\,\mathrm{d}x =} \\
&\leq E(u) - E_\ell(u_\ell) + C_2 \operatorname{osc}(f, \mathcal{T}_\ell) + \int_\Omega (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : \mathrm{D}u\,\mathrm{d}x
\end{aligned}
$$

LEB

$$
\begin{aligned}
0 &\le \int_\Omega \big(W(\mathrm{D}u) - W(\mathcal{G}_\ell u_\ell) - \mathrm{D}W(\mathcal{G}_\ell u_\ell) : (\mathrm{D}u - \mathcal{G}_\ell u_\ell)\big)\,\mathrm{d}x \\
&= \int_\Omega \big(W(\mathrm{D}u) - W(\mathcal{G}_\ell u_\ell) + (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : (\mathrm{D}u - \mathcal{G}_\ell u_\ell)\big)\,\mathrm{d}x - \underbrace{\int_\Omega \sigma_\ell : (\mathrm{D}u - \mathcal{G}_\ell u_\ell)\,\mathrm{d}x}_{\int_\Omega \Pi^k_{\mathcal{T}_\ell} f\cdot(u - u_{\mathcal{T}_\ell})\,\mathrm{d}x =} \\
&\le E(u) - E_\ell(u_\ell) + C_2 \operatorname{osc}(f, \mathcal{T}_\ell) + \int_\Omega (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : \mathrm{D}u\,\mathrm{d}x
\end{aligned}
$$

$$
\mathrm{LEB}_\ell := E_\ell(u_\ell) - C_2 \operatorname{osc}(f, \mathcal{T}_\ell) - \textstyle\int_\Omega (\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)) : \mathrm{D}u\,\mathrm{d}x \le E(u)
$$

Convergence of $E_\ell(u_\ell)$

Weak lower semicontinuity and $\lim_{\ell\to\infty} \mathrm{osc}(f, \mathcal{T}_\ell) = 0 \implies$

$$E(v) \le \liminf_{\ell\to\infty} \int_\Omega (W(\mathcal{G}_\ell u_\ell) - f \cdot \mathcal{J}_\ell u_\ell)\,\mathrm{d}x$$
$$= \liminf_{\ell\to\infty} (E_\ell(u_\ell) + \int_\Omega f \cdot (1 - \Pi^k_{\mathcal{T}_\ell})\mathcal{J}_\ell u_\ell) = \liminf_{\ell\to\infty} \mathrm{LEB}_\ell \le E(u)$$

$\implies \lim_{\ell\to\infty} E_\ell(u_\ell) = \lim_{\ell\to\infty} \mathrm{LEB}_\ell = E(u)$ $\square$

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$\implies \lim_{\ell\to\infty} E_\ell(u_\ell) = \lim_{\ell\to\infty} \mathrm{LEB}_\ell = E(u)$ $\square$

If (cc), then $\|\sigma - \mathrm{D}W(\mathcal{G}_\ell u_\ell)\|^2_{L^{p'}(\Omega)} \lesssim E(u) - \mathrm{LEB}_\ell \to 0$ $\square$

If W strongly convex, then $\|\mathrm{D}u - \mathcal{G}_\ell u_\ell\|^p_{L^p(\Omega)} \lesssim E(u) - \mathrm{LEB}_\ell \to 0$ $\square$

Lavrentiev gap

$$\inf E(V) < \inf E(W_0^{1,\infty}(\Omega;\mathbb{R}^m))$$

- $W \in C^1(\mathbb{M})$, $c_1|A|^p - c_2 \le W(A)\ \forall A \in \mathbb{M}$
- no upper growth $\implies$ $\sigma \notin L^{p'}(\Omega;\mathbb{M})$ and ELE not well-defined in general
- $\lim_{\ell\to\infty} \|\sigma_\ell - \mathrm{D}W(\mathcal{G}_\ell u_\ell)\|_{L^{p'}(\Omega)} = \infty$ may hold $\implies$ no convergence of $\eta_\ell^{(\varepsilon)}$
- Plain convergence for $k = 0$ possible because of LEB

$$E_\ell(u_\ell) - C\|h_\ell f\|_{L^{p'}(\Omega)} \le \min E(u)$$

Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

C and Ortner: Analysis of a class of penalty methods for computing singular minimizers. CMAM (2010)

4-Laplace on L-shaped domain

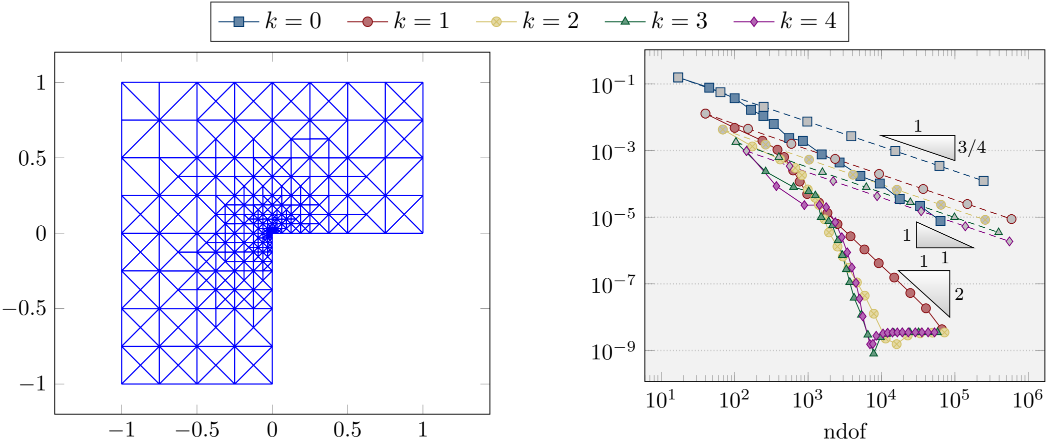
Minimize $\displaystyle\int_\Omega (W(\mathrm{D}v) - fv)\,\mathrm{d}x - \int_{\Gamma_\mathrm{N}} gv\,\mathrm{d}s$

- $W(a) := |a|^4/4\ \forall a \in \mathbb{R}^2$
- $\Omega := (-1,1)^2 \setminus ([0,1) \times (-1,0])$
- $f(r,\varphi) := 343/2048 r^{-11/8} \sin(7\varphi/8)$
- $u_\mathrm{D}(r,\varphi) := r^{7/8} \sin(7\varphi/8)$ on $\Gamma_\mathrm{D} := (0 \times [-1,0]) \cup ([0,1] \times 0)$
- $g(r,\varphi) := 343/512 r^{-3/8}(-\sin(\varphi/8), \cos(\varphi/8)) \cdot \nu$ on $\Gamma_\mathrm{N} := \partial\Omega \setminus \Gamma_\mathrm{D}$

Parameter choice: $\varepsilon = (k+1)/100$, $\theta = 0.5$

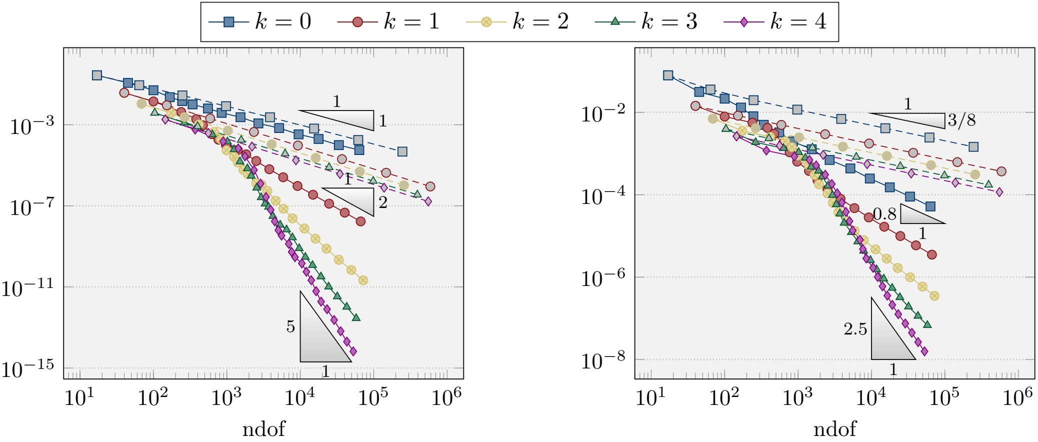
C and Klose: A posteriori finite element error control for the p-Laplace problem. SINUM (2003)

4-Laplace on L-shaped domain (cont.)



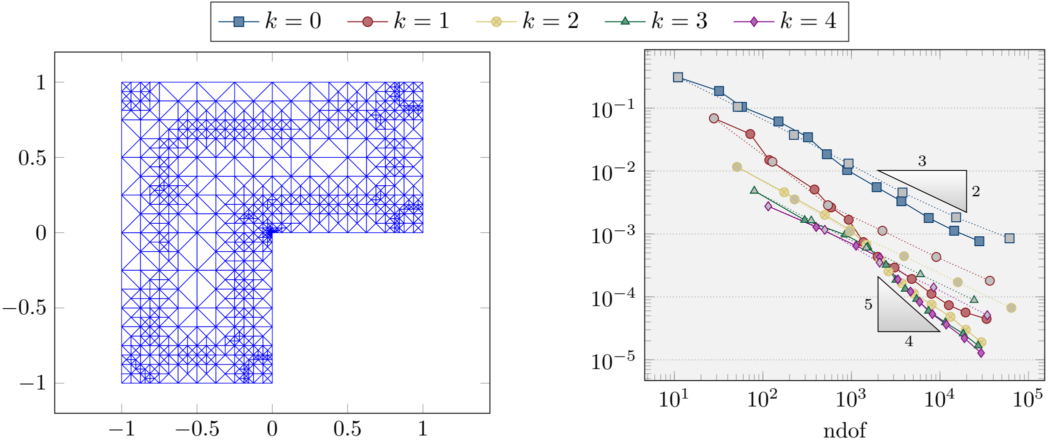
Adaptive triangulation and convergence history plot of $|E(u) - E_h(u_h)|$ (right)

4-Laplace on L-shaped domain (cont.)



Convergence history plot of $\|\sigma - \mathrm{D}W(\mathcal{G}u_h)\|^2_{L^{4/3}(\Omega)}$ (left) and $\|\mathrm{D}u - \mathcal{G}u_h\|^2_{L^4(\Omega)}$ (right)

Optimal design problem on L-shaped domain ($f \equiv 1$, $u_D \equiv 0$ on $\partial\Omega$)



Adaptive triangulation and convergence history plot of RHS (right)

Modified Foss-Hrusa-Mizel benchmark

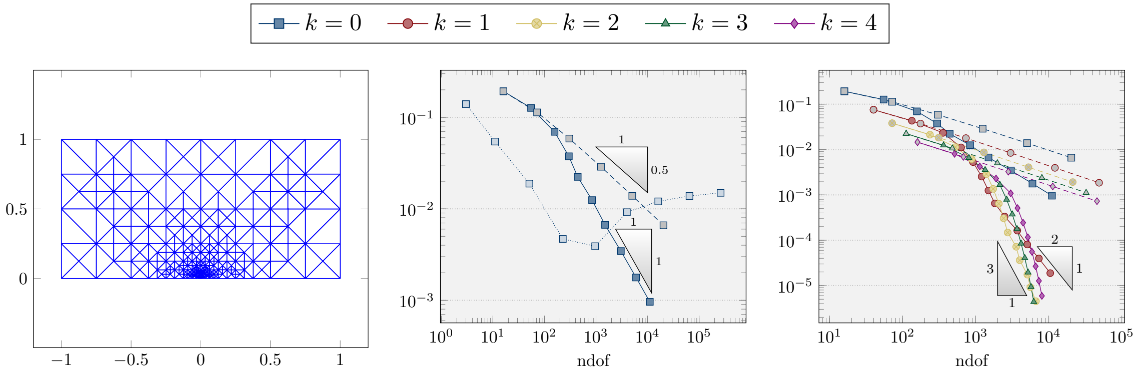
Minimize $E(v) := \int_\Omega W(\mathrm{D}v)\,\mathrm{d}x$ among $v \in \mathcal{A}$

- $W(A) := (|A|^2 - 2\det A)^4 + |A|^2/2$ for all $A \in \mathbb{M} := \mathbb{R}^{2\times 2}$
- $\Omega := (-1,1) \times (0,1)$
- $\Gamma_1 := [-1,0] \times \{0\}$, $\Gamma_2 := [0,1] \times \{0\}$, $\Gamma_3 := \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$
- $u_\mathrm{D} := (\cos(\varphi/2), \sin(\varphi/2))$ in polar coordinates
- $\mathcal{A} := \{v = (v_1, v_2) \in W^{1,2}(\Omega;\mathbb{R}^2) : v_1 \equiv 0 \text{ on } \Gamma_1, v_2 \equiv 0 \text{ on } \Gamma_2, v = u_\mathrm{D} \text{ on } \Gamma_3\}$

Foss, Hrusa, Mizel: The Lavrentiev gap phenomenon in nonlinear elasticity. Arch. Ration. Mech. Anal. (2003)

Ortner and Praetorius: On the convergence of adaptive nonconforming finite element methods for a class of convex variational problems. SINUM (2011)

Modified Foss-Hrusa-Mizel benchmark (cont.)



Adaptive triangulation (left), numerical evidence for Lavrentiev gap (middle), and convergence history plot of $|E(u) - E_h(u_h)|$ (right)

Conclusions

- a priori + a posteriori error estimates
- convergence rates
- super-linear convergent LEB
- convergent adaptive scheme
- improved empirical convergence rates for larger k and adaptive mesh refinements

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Thank you for your attention

C and Tran: Unstabilized hybrid high-order method for a class of degenerate convex minimization problems. SINUM (2021)

C and Tran: Convergent adaptive hybrid higher-order schemes for convex minimization. Numer. Math. (2022)