Robust least squares methods for the Helmholtz equation

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MINRES/LS-5, Oct. 5-7, 2022

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Outline

1. Ultra-weak first order system formulation
   - Well-posedness
   - MINRES: Residual minimization using optimal test norm
   - ‘Practical’ MINRES
   - ‘Boosted’ (practical) MINRES
2. Numerical results
3. Inf-sup stability unif. in h and κ (WIP)
4. References

Helmholtz equation

Let $\Omega \subset \mathbb{R}^d$ bounded Lip., with $\partial\Omega = \Gamma_D \dot\cup \Gamma_N \dot\cup \Gamma_I$, where $|\Gamma_I| > 0$, and $\kappa > 0$. For $f \in H^1_{0,\Gamma_D}(\Omega)'$, $g_D \in H^{\frac{1}{2}}(\Gamma_D)$, and $g \in H^{-\frac{1}{2}}(\Gamma_N \cup \Gamma_I)$, find $\phi \in H^1(\Omega)$ s.t.

$$\begin{aligned} -\Delta\phi - \kappa^2\phi &= \kappa^2 f \quad \text{on } \Omega, & \phi &= \kappa g_D \quad \text{on } \Gamma_D, \\ \tfrac{\partial\phi}{\partial\vec{n}} &= \kappa^2 g \quad \text{on } \Gamma_N, & \tfrac{\partial\phi}{\partial\vec{n}} \pm i\kappa\phi &= \kappa^2 g \quad \text{on } \Gamma_I, \end{aligned} \tag{1}$$

with '$\pm$' either '$+$' or '$-$'.

Standard var. form.: Find $\phi \in H^1_{\kappa g_D,\Gamma_D}(\Omega) := \{\breve\phi \in H^1(\Omega) : \breve\phi|_{\Gamma_D} = \kappa g_D \text{ on } \Gamma_D\}$ such that

$$\begin{aligned} (L_\kappa\phi)(\eta) &:= \int_\Omega \nabla\phi \cdot \nabla\bar\eta - \kappa^2\phi\bar\eta \, dx \pm i\kappa \int_{\Gamma_I} \phi\bar\eta \, ds \\ &= f(v) + \int_{\Gamma_N \cup \Gamma_I} \kappa^2 g\bar\eta \, ds \quad (\eta \in H^1_{0,\Gamma_D}(\Omega)). \end{aligned} \tag{2}$$

It is known, e.g. [Ern and Guermond, 2021], that

$$L_\kappa \in \mathcal{L}\mathrm{is}(H^1_{0,\Gamma_D}(\Omega), H^1_{0,\Gamma_D}(\Omega)') \quad [1] \tag{3}$$

(*not* uniform in κ).

[1] Same holds true when $|\Gamma_I| = 0$ and κ^2 is not an eigenvalue of the Laplace operator with (homogeneous) mixed Dirichlet/Neumann boundary conditions.

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For $f \in H^1_{0,\Gamma_D}(\Omega)'$, $g_D \in H^{\frac12}(\Gamma_D)$, and $g \in H^{-\frac12}(\Gamma_N \cup \Gamma_I)$, find $\phi \in H^1(\Omega)$ s.t.

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Standard finite element Galerkin

- Sesquilinear form $(L_\kappa\phi)(\eta)$ is Hermitian, but not coercive (unless $\kappa c_{\text{Poincaré}} < 1$), and Galerkin solution may not exist on a coarse mesh. When it exists, generally it is not quasi-optimal (*pollution*).
- [Melenk and Sauter, 2011]: Polygonal domain Ω, with a quasi-uniform mesh (geometrically refined near the vertices). Then quasi-optimality w.r.t. $\sqrt{\|\cdot\|^2_{L_2(\Omega)} + \frac{1}{\kappa^2}\|\nabla\cdot\|^2_{L_2(\Omega)}}$ when $\frac{\kappa h}{p}$ and $\frac{\log\kappa}{p}$ are small enough. (Classical condition for the h-method is $\kappa^2 h$ small enough).
- Question: **Is it possible to obtain quasi-best approximations from fem-space without such conditions?**
- κh small enough is needed anyway to obtain an acceptable (best) approximation error. Indeed, since for $\|\vec{r}\| = 1$, $\vec{x} \mapsto e^{-i\kappa\vec{r}\cdot\vec{x}} \in \ker(-\Delta - \kappa^2\mathrm{Id})$, typically solution ϕ of Helmholtz has large components in the direction of such plane waves whose wave length is $\frac{2\pi}{\kappa}$. (Consequently, as a singularly perturbed problem, Helmholtz is more benign than say convection dominated convection diffusion).

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First order system

Following [Demkowicz, Gopalakrishnan, Muga, and Zitelli, 2012] we consider an *ultra-weak* formulation. First, we rewrite Helmholtz as *1st order system*:

We decompose f as

$$f(\eta) = \int_\Omega f_1 \overline{\eta} + \tfrac{1}{\kappa} \vec{f}_2 \cdot \nabla \overline{\eta} \, dx \quad (\eta \in H^1_{0,\Gamma_D}(\Omega)),$$

for some $f_1 \in L_2(\Omega)$ and $\vec{f}_2 \in L_2(\Omega)^d$.

Setting $\vec{u} = \frac{1}{\kappa} \nabla \phi - \vec{f}_2$ we arrive at

$$\begin{aligned}
-\tfrac{1}{\kappa} \nabla \cdot \vec{u} - \phi &= f_1 && \text{on } \Omega, \\
\tfrac{1}{\kappa} \nabla \phi - \vec{u} &= \vec{f}_2 && \text{on } \Omega, \\
\phi &= \kappa g_D && \text{on } \Gamma_D, \\
\vec{u} \cdot \vec{n} &= \kappa g && \text{on } \Gamma_N, \\
\vec{u} \cdot \vec{n} \pm i\phi &= \kappa g && \text{on } \Gamma_I.
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(global) ultra weak

Testing 1st and 2nd eq. with η and $\vec{v}$, for smooth η and $\vec{v}$ with

- $\eta = 0$ on Γ_D, $\vec{v} \cdot \vec{n} = 0$ on Γ_N, $\vec{v} \cdot \vec{n} \mp i\eta = 0$ on Γ_I,

i.e. (hom.) (essential) (adjoint) b.c., and applying int-by-parts

$$\big(B_\kappa(\phi, \vec{u})\big)(\eta, \vec{v}) := \int_\Omega \tfrac{1}{\kappa}\vec{u} \cdot \nabla\overline{\eta} - \phi\overline{\eta} - \tfrac{1}{\kappa}\phi\, \nabla \cdot \overline{\vec{v}} - \vec{u} \cdot \overline{\vec{v}}\, dx$$

$$= \int_\Omega f_1\overline{\eta} + \vec{f}_2 \cdot \overline{\vec{v}}\, dx - \int_{\Gamma_D} g_D\overline{\vec{v}} \cdot \vec{n}\, ds + \int_{\Gamma_N \cup \Gamma_I} g\overline{\eta}\, ds =: q(\eta, \vec{v}).$$

All boundary conditions are *natural*.

Well-posedness I

Theorem

With $U := L_2(\Omega) \times L_2(\Omega)^d$ *and*

$$V_{\mp} := \Big\{ (\eta, \vec{v}) \in H^1_{0,\Gamma_D}(\Omega) \times H(\mathrm{div};\Omega) : \int_{\Gamma_N \cup \Gamma_I} \vec{v} \cdot \vec{n}\, \overline{\psi}\, ds \mp i \int_{\Gamma_I} \eta \overline{\psi}\, ds = 0 \quad (\psi \in H^1_{0,\Gamma_D}(\Omega)) \Big\}$$

equipped with canonical norms, $B_\kappa \in \mathcal{L}\mathrm{is}(U, V'_{\mp})$ *(not uniform in* κ*).*

Well-posedness II

Proof.

Boundedness: ✓.

Injectivity: $(\phi, \vec{u}) \in \ker B_\kappa$. $\big(B_\kappa(\phi, \vec{u})\big)(\eta, \vec{v}) = 0$ for test functions $(\eta, \vec{v})$ gives

$$\nabla \cdot \vec{u} + \kappa\phi = 0, \quad \nabla\phi - \kappa\vec{u} = 0, \tag{5}$$

in part. $(\phi, \vec{u}) \in H^1(\Omega) \times H(\text{div}; \Omega)$. Int-by-parts for $(\eta, \vec{v}) \in V_\mp$ gives $(\ldots)$

$$\phi \in H^1_{0,\Gamma_D}(\Omega), \quad \int_{\partial\Omega} \vec{u} \cdot \vec{n}\,\overline{\eta}\,ds \pm i \int_{\Gamma_I} \phi\overline{\eta}\,ds = 0 \quad (\eta \in H^1_{0,\Gamma_D}(\Omega)).$$

For $\eta \in H^1_{0,\Gamma_D}(\Omega)$,

$$\begin{aligned}(L_\kappa\phi)(\eta) &= \int_\Omega \nabla\phi \cdot \nabla\overline{\eta} - \kappa^2\phi\overline{\eta}\,dx \pm i\kappa \int_{\Gamma_I} \phi\overline{\eta}\,ds \\ &= \int_\Omega \kappa\vec{u} \cdot \nabla\overline{\eta} - \kappa^2\phi\overline{\eta}\,dx - \kappa \int_{\partial\Omega} \vec{u} \cdot \vec{n}\,\overline{\eta}\,ds \\ &= -\kappa \int_\Omega (\nabla \cdot \vec{u} + \kappa\phi)\overline{\eta}\,dx = 0\end{aligned}$$

so $\phi = 0$, and so $\vec{u} = 0$.

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For $\eta \in H^1_{0,\Gamma_D}(\Omega)$,

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in part. $(\phi, \vec{u}) \in H^1(\Omega) \times H(\mathrm{div}; \Omega)$. Int-by-parts for $(\eta, \vec{v}) \in V_\mp$ gives $(\ldots)$

$$\phi \in H^1_{0,\Gamma_D}(\Omega), \quad \int_{\partial\Omega} \vec{u} \cdot \vec{n}\,\overline{\eta}\,ds \pm i \int_{\Gamma_I} \phi\overline{\eta}\,ds = 0 \quad (\eta \in H^1_{0,\Gamma_D}(\Omega)).$$

For $\eta \in H^1_{0,\Gamma_D}(\Omega)$,

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Well-posedness III

Proof (cont'd).

Surjectivity: Let $q \in V'_{\mp}$. Riesz: $\exists(\zeta, \vec{w}) \in V_{\mp}$ with

$$q(\eta, \vec{v}) = \int_\Omega \vec{w} \cdot \overline{\vec{v}} + \nabla \cdot \vec{w}\, \nabla \cdot \overline{\vec{v}} + \zeta \overline{\eta} + \nabla \zeta \cdot \nabla \overline{\eta}\, dx \quad ((\eta, \vec{v}) \in V_{\mp}).$$

With $\underset{\sim}{\vec{u}} = \vec{u} + \vec{w}$, $\underset{\sim}{\phi} = \phi + \kappa \nabla \cdot \vec{w}$, $B_\kappa(\phi, \vec{u}) = q \iff$

$$\begin{aligned} \int_\Omega \tfrac{1}{\kappa} \underset{\sim}{\vec{u}} \cdot \nabla \overline{\eta} - \underset{\sim}{\phi} \overline{\eta} - \tfrac{1}{\kappa} \underset{\sim}{\phi}\, \nabla \cdot \overline{\vec{v}} - \underset{\sim}{\vec{u}} \cdot \overline{\vec{v}}\, dx = \\ \int_\Omega (\zeta - \kappa \nabla \cdot \vec{w}) \overline{\eta} + (\nabla \zeta + \tfrac{\vec{w}}{\kappa}) \cdot \nabla \overline{\eta}\, dx =: \underset{\sim}{q}(\eta) \quad ((\eta, \vec{v}) \in V_{\mp}), \end{aligned} \tag{6}$$

where $\underset{\sim}{q} \in H^1_{0,\Gamma_D}(\Omega)'$. Given $\underset{\sim}{\phi} \in H^1_{0,\Gamma_D}(\Omega)$, let $\underset{\sim}{\vec{u}} := \frac{1}{\kappa} \nabla \underset{\sim}{\phi}$. Then (6) $\iff$

$$\int_\Omega \nabla \underset{\sim}{\phi} \cdot \nabla \overline{\eta} - \kappa^2 \underset{\sim}{\phi} \overline{\eta} - \kappa(\underset{\sim}{\phi} \nabla \cdot \overline{\vec{v}} + \nabla \underset{\sim}{\phi} \cdot \overline{\vec{v}})\, dx = \kappa^2 \underset{\sim}{q}(\eta) \quad ((\eta, \vec{v}) \in V_{\mp}). \tag{7}$$

Since $\int_\Omega \underset{\sim}{\phi} \cdot \nabla \cdot \overline{\vec{v}} + \nabla \underset{\sim}{\phi} \cdot \overline{\vec{v}}\, dx = \int_{\partial\Omega} \underset{\sim}{\phi} \overline{\vec{v}} \cdot \vec{n}\, dx = \mp i \int_{\Gamma_I} \underset{\sim}{\phi} \overline{\eta}\, ds$ by definition of $V_{\mp}$, (7) $\iff$ $L_\kappa \underset{\sim}{\phi} = \kappa^2 \underset{\sim}{q}$, and so $\underset{\sim}{\phi} \in H^1_{0,\Gamma_D}(\Omega)$ exists.

Well-posedness III

Proof (cont'd).

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With $\underset{\sim}{\vec{u}} = \vec{u} + \vec{w}$, $\underset{\sim}{\phi} = \phi + \kappa\nabla \cdot \vec{w}$, $B_\kappa(\phi, \vec{u}) = q \iff$

$$\begin{aligned}\int_\Omega \tfrac{1}{\kappa}\underset{\sim}{\vec{u}} \cdot \nabla\overline{\eta} - \underset{\sim}{\phi}\overline{\eta} - \tfrac{1}{\kappa}\underset{\sim}{\phi}\,\nabla \cdot \overline{\vec{v}} - \underset{\sim}{\vec{u}} \cdot \overline{\vec{v}}\, dx = \\ \int_\Omega (\zeta - \kappa\nabla \cdot \vec{w})\overline{\eta} + (\nabla\zeta + \tfrac{\vec{w}}{\kappa}) \cdot \nabla\overline{\eta}\, dx =: \underset{\sim}{q}(\eta) \quad ((\eta, \vec{v}) \in V_\mp),\end{aligned} \tag{6}$$

where $\underset{\sim}{q} \in H^1_{0,\Gamma_D}(\Omega)'$. Given $\underset{\sim}{\phi} \in H^1_{0,\Gamma_D}(\Omega)$, let $\underset{\sim}{\vec{u}} := \frac{1}{\kappa}\nabla\underset{\sim}{\phi}$. Then (6) $\iff$

$$\int_\Omega \nabla\underset{\sim}{\phi} \cdot \nabla\overline{\eta} - \kappa^2\underset{\sim}{\phi}\overline{\eta} - \kappa(\underset{\sim}{\phi}\nabla \cdot \overline{\vec{v}} + \nabla\underset{\sim}{\phi} \cdot \overline{\vec{v}})\, dx = \kappa^2\underset{\sim}{q}(\eta) \quad ((\eta, \vec{v}) \in V_\mp). \tag{7}$$

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MINRES with optimal test norm

To solve

$$(B_\kappa(\phi,\vec{u}))(\eta,\vec{v}) = q(\eta,\vec{v}) \quad ((\eta,\vec{v}) \in V_\mp).$$

$B_\kappa \in \mathcal{L}\mathrm{is}(U, V'_\mp)$, $U \simeq U'$, so $B'_\kappa \in \mathcal{L}\mathrm{is}(V_\mp, U)$. Aiming at "robustness", equip $V_\mp$ with *optimal test norm*

$$\|(\eta,\vec{v})\|_{V_{\mp,\kappa}} := \|B'_\kappa(\eta,\vec{v})\|_U = \sqrt{\|\tfrac{1}{\kappa}\nabla\cdot\vec{v}+\eta\|^2_{L_2(\Omega)} + \|\tfrac{1}{\kappa}\nabla\eta - \vec{v}\|^2_{L_2(\Omega)^d}},$$

and so $V'_\mp$ with associated dual norm $\|\cdot\|_{V'_{\mp,\kappa}}$.

With $\mathbb{u} := (\phi,\vec{u})$, $\mathbb{v} := (\eta,\vec{v})$,

$$\|B_\kappa \mathbb{u}\|_{V'_{\mp,\kappa}} = \sup_{0\neq\mathbb{v}\in V_\mp} \frac{|(B_\kappa\mathbb{u})(\mathbb{v})|}{\|B'_\kappa\mathbb{v}\|_U} = \sup_{0\neq\mathbb{v}\in V_\mp} \frac{|\langle \mathbb{u}, B'_\kappa\mathbb{v}\rangle_U|}{\|B'_\kappa\mathbb{v}\|_U} = \|\mathbb{u}\|_U,$$

i.e., $B_\kappa \in \mathcal{L}\mathrm{is}(U, V'_\mp)$ is *isometry*. For any closed subspace $U^\delta \subset U$,

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is *best* approximation to $\mathbb{u}$ from U^δ.

MINRES with optimal test norm

To solve

$$(B_\kappa(\phi,\vec{u}))(\eta,\vec{v}) = q(\eta,\vec{v}) \quad ((\eta,\vec{v})\in V_\mp).$$

$B_\kappa \in \mathcal{L}\mathrm{is}(U, V'_\mp)$, $U \simeq U'$, so $B'_\kappa \in \mathcal{L}\mathrm{is}(V_\mp, U)$. Aiming at "robustness", equip $V_\mp$ with *optimal test norm*

$$\|(\eta,\vec{v})\|_{V_{\mp,\kappa}} := \|B'_\kappa(\eta,\vec{v})\|_U = \sqrt{\|\tfrac{1}{\kappa}\nabla\cdot\vec{v}+\eta\|^2_{L_2(\Omega)} + \|\tfrac{1}{\kappa}\nabla\eta-\vec{v}\|^2_{L_2(\Omega)^d}},$$

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is *best* approximation to $\mathbb{u}$ from U^δ.

Practical method

$\mathbb{u}^\delta := \underset{\mathbb{w}^\delta \in U^\delta}{\operatorname{argmin}} \|q - B_\kappa \mathbb{w}^\delta\|_{V'_{\mp,\kappa}}$ is 2nd comp. of $(\mathbb{v}, \mathbb{u}^\delta) \in V_\mp \times U^\delta$ that solves

$$\begin{cases} \langle B'_\kappa \mathbb{v}, B'_\kappa \underset{\sim}{\mathbb{v}} \rangle_U + \langle \mathbb{u}^\delta, B'_\kappa \underset{\sim}{\mathbb{v}} \rangle_U & = & q(\underset{\sim}{\mathbb{v}}) & (\underset{\sim}{\mathbb{v}} \in V_\mp), \\ \langle B'_\kappa \mathbb{v}, \underset{\sim}{\mathbb{u}}^\delta \rangle_U & = & 0 & (\underset{\sim}{\mathbb{u}}^\delta \in U^\delta). \end{cases}$$

Replacing $V_\mp$ by a closed subspace $V_\mp^\delta$ s.t.

$$\gamma^\delta := \inf_{0 \neq \underset{\sim}{\mathbb{u}}^\delta \in U^\delta} \sup_{0 \neq \underset{\sim}{\mathbb{v}}^\delta \in V_\mp^\delta} \frac{\langle \underset{\sim}{\mathbb{u}}^\delta, B'_\kappa \underset{\sim}{\mathbb{v}}^\delta \rangle_U}{\|\underset{\sim}{\mathbb{u}}^\delta\|_U \|B'_\kappa \underset{\sim}{\mathbb{v}}^\delta\|_U} > 0,$$

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Discussion

- MINRES with optimal test norm introduced in [Barrett and Morton, 1984, Demkowicz and Gopalakrishnan, 2011, Cohen, Dahmen, and Welper, 2012].
- Optimal test norm is computable because trial space $U = L_2(\Omega)^{d+1}$, thanks to ultra-weak formulation. Test space $V_\mp$ 'involves' only derivatives of 1st order, because ultra-weak was derived from a system of 1st order.
- In [Demkowicz, Gopalakrishnan, Muga, and Zitelli, 2012] 1st order system was tested and integrated-by-parts element-wise (DPG). Besides the "field variables" $(\phi, \vec{u}) \in L_2(\Omega)^{d+1}$, it gives additional "trace variables" $(\hat{\phi}, \hat{u}_n)$ on the mesh-skeleton $\partial\mathcal{T}$ with (quotient) norms $\|\hat{\phi}\| = \inf\{\|\phi\|_{H^1(\Omega)} : \phi|_{\partial\mathcal{T}} = \hat{\phi}\}$, $\|\hat{u}_n\| = \inf\{\|\vec{u}\|_{H(\mathrm{div};\Omega)} : \vec{u}|_{\partial\mathcal{T}} \cdot \vec{n} = \hat{u}_n\}$. Possible quasi-optimality will be for the tuple $(\phi, \vec{u}, \hat{\phi}, \hat{u}_n)$. One higher order approximation is applied for the trace variables. A priori bounds that demonstrate a certain rate involve derivatives of solution of one higher order.

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Discussion cont'd

- Advantages of DPG include
  - System matrix w.r.t. scalar product on $V_{\mp} \times V_{\mp}$ of $V_{\mp}^{\delta}$ is block diagonal, and 'practical DPG' solution can be computed by inverting SPD system. ('opt. test functions' are local).
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Boosted MINRES

Recall

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Booster method yields $\mathbb{u}^\delta + B'_\kappa \mathbb{v}^\delta$ being, modulo factor $1/\gamma^\delta$, best approx. to $\mathbb{u}$ from $U^\delta + B'_\kappa V^\delta_\mp$. Altern. compute $\mathbb{v}^\delta \in V^\delta_\mp$ from

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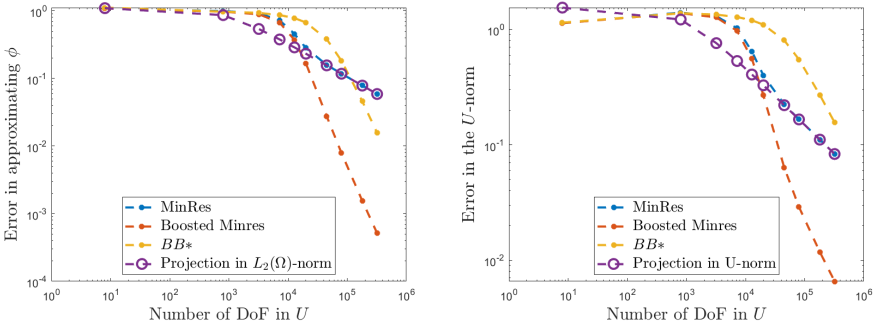
Numerical results

$\Omega = (0,1)^2$, $\Gamma_I = \partial\Omega$, $\phi(\vec{x}) = e^{i\kappa\vec{r}\cdot\vec{x}}$, $\vec{r} = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$, $\kappa = 100$,
$f = f_1$, so $\vec{u} = \kappa^{-1}\nabla\phi$ (in all examples).

$U^\delta = \mathcal{S}^{-1,0}_{\mathcal{T}^\delta} \times (\mathcal{S}^{-1,0}_{\mathcal{T}^\delta})^2$, $V^\delta_\mp = (\mathcal{S}^{0,2}_{\mathcal{T}^\delta} \times RT_{2,\mathcal{T}^\delta}) \cap V_\mp$ (degree +2 in all ex.s).



All computations using NGSolve.

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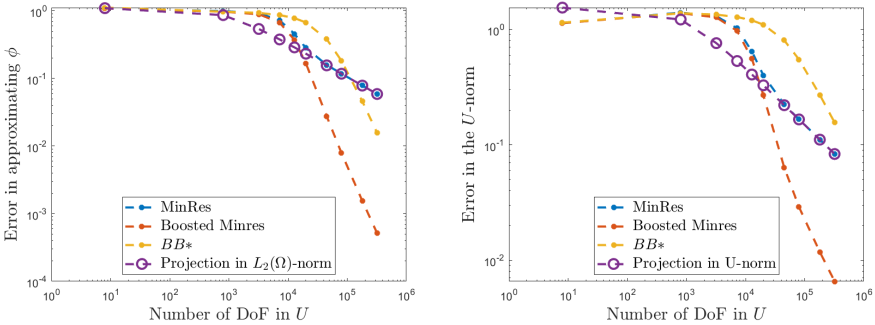

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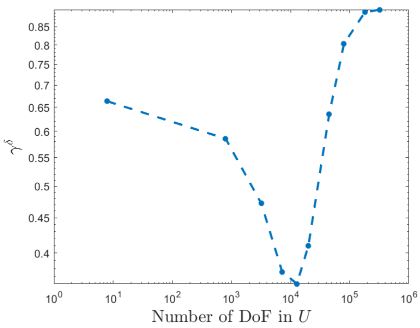
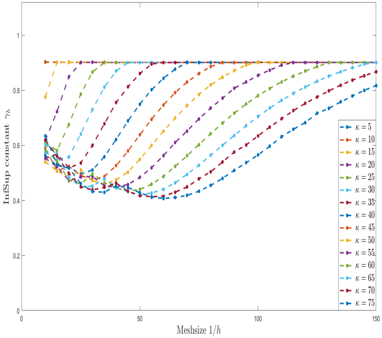
Numerical results: γ^δ

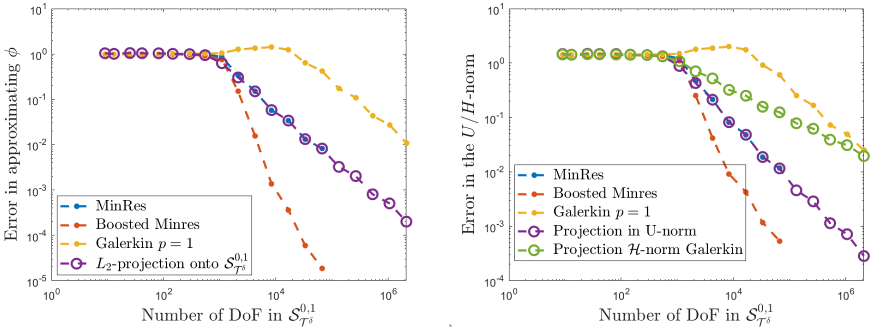




Numerical results: Comparison with plain Galerkin

Same problem. $U^\delta = \mathcal{S}^{0,1}_{\mathcal{T}^\delta} \times (\mathcal{S}^{0,1}_{\mathcal{T}^\delta})^2$, $V^\delta_\mp = (\mathcal{S}^{0,3}_{\mathcal{T}^\delta} \times RT_{3,\mathcal{T}^\delta}) \cap V_\mp$.

Galerkin with $X^\delta = \mathcal{S}^{0,1}_{\mathcal{T}^\delta}$.



Since $\vec{u} = \frac{1}{\kappa}\nabla\phi$, $\|\cdot\|_U$-error in approx. for $(\phi, \vec{u})$ is $\|\cdot\|_{\mathcal{H}}$-error in approx. for ϕ, where $\|\cdot\|_{\mathcal{H}} := \sqrt{\|\cdot\|^2_{L_2(\Omega)} + \frac{1}{\kappa^2}\|\nabla\cdot\|^2_{L_2(\Omega)}}$.

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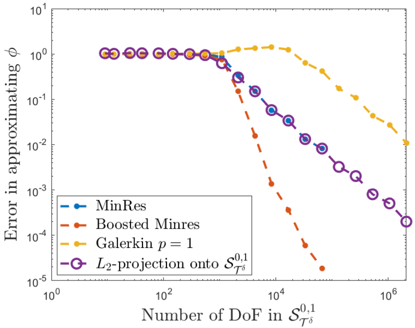
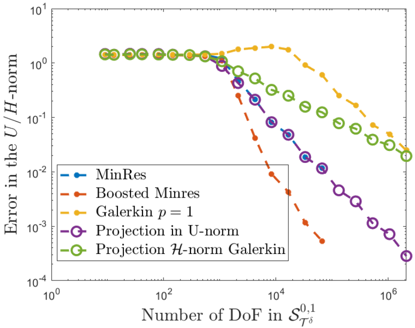
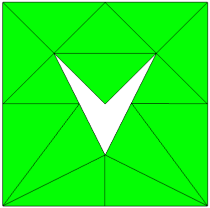




Since $\vec{u} = \frac{1}{\kappa}\nabla\phi$, $\|\cdot\|_U$-error in approx. for $(\phi, \vec{u})$ is $\|\cdot\|_{\mathcal{H}}$-error in approx. for ϕ, where $\|\cdot\|_{\mathcal{H}} := \sqrt{\|\cdot\|^2_{L_2(\Omega)} + \frac{1}{\kappa^2}\|\nabla\cdot\|^2_{L_2(\Omega)}}$.

Numerical results: Scattering problem

Domain Ω and initial mesh:

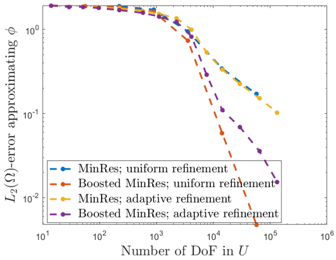


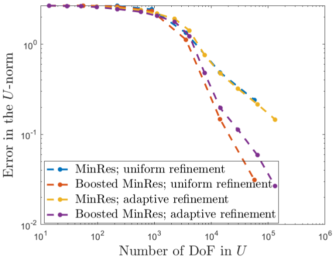
$\kappa = 10\pi$. Hom. Dir. on inner brd, impedance on outer bdr with $\kappa^2 g = \frac{\partial \phi_{\text{in}}}{\partial \vec{n}} \pm i\kappa\phi_{\text{in}}$ and $\phi_{\text{in}}(\vec{x}) = e^{i\kappa\vec{r}\cdot\vec{x}}$, $\vec{r} = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$.

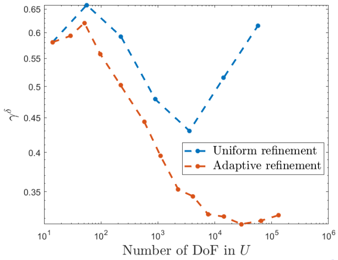
(This and next example from [Chaumont-Frelet, Ern, and Vohralík, 2021]).

Numerical results: Scattering problem lowest order

$U^\delta = \mathcal{S}^{-1,0}_{\mathcal{T}^\delta} \times (\mathcal{S}^{-1,0}_{\mathcal{T}^\delta})^2$, $V^\delta_\mp = (\mathcal{S}^{0,2}_{\mathcal{T}^\delta} \times RT_{2,\mathcal{T}^\delta}) \cap V_\mp$. Dörfler $\theta = 0.6$.

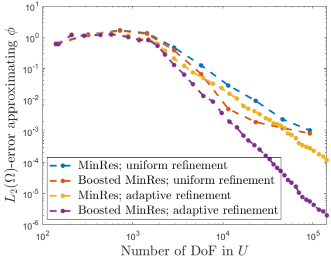


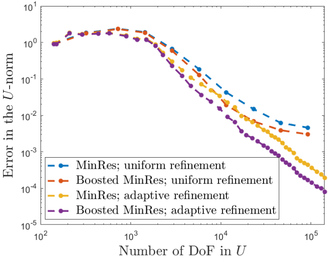


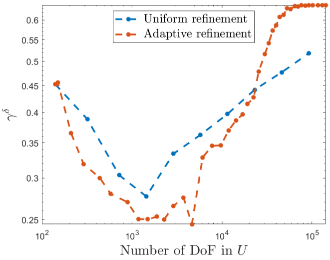


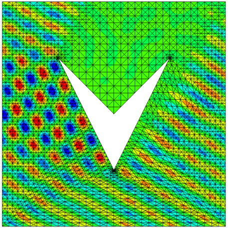
Numerical results: Scattering problem cubics

$$U^\delta = \mathcal{S}^{-1,3}_{\mathcal{T}^\delta} \times (\mathcal{S}^{-1,3}_{\mathcal{T}^\delta})^2, \; V^\delta_\mp = (\mathcal{S}^{0,5}_{\mathcal{T}^\delta} \times RT_{5,\mathcal{T}^\delta}) \cap V_\mp.$$







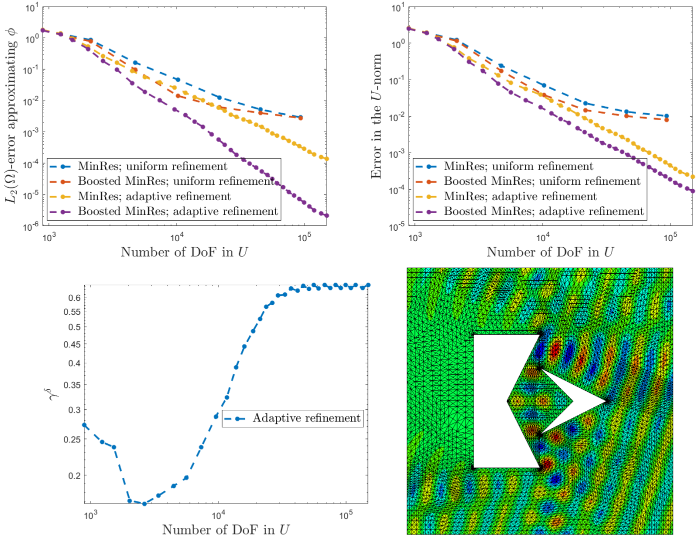


Numerical results: Trapping problem cubics

$\kappa = 10\pi$. Hom. Dir. on inner brd, impedance on outer bdr with $\kappa^2 g = \frac{\partial \phi_{\text{in}}}{\partial \vec{n}} \pm i\kappa\phi_{\text{in}}$ and $\phi_{\text{in}}(\vec{x}) = e^{i\kappa\vec{r}\cdot\vec{x}}$, $\vec{r} = (\cos\frac{9\pi}{10}, \sin\frac{9\pi}{10})$.

$U^\delta = \mathcal{S}^{-1,3}_{\mathcal{T}^\delta} \times (\mathcal{S}^{-1,3}_{\mathcal{T}^\delta})^2$, $V^\delta_\mp = (\mathcal{S}^{0,5}_{\mathcal{T}^\delta} \times RT_{5,\mathcal{T}^\delta}) \cap V_\mp$.

Numerical results: Trapping problem cubics



Inf-sup stability

To show $\gamma^\delta := \inf_{0\neq \mathbb{u}^\delta \in U^\delta} \sup_{0\neq \mathbb{v}^\delta \in V_\mp^\delta} \frac{\langle \mathbb{u}^\delta, B_\kappa' \mathbb{v}^\delta\rangle_U}{\|\mathbb{u}^\delta\|_U \|B_\kappa' \mathbb{v}^\delta\|_U} \gtrsim 1$ unif. in "δ", and κ.

If ${B_\kappa'}^{-1}\mathbb{u}^\delta \in V_\mp$ would be in $V_\mp^\delta$, then $\gamma^\delta = 1$. So with $P^\delta : V_\mp \to V_\mp^\delta$ orth. proj. w.r.t. $\|\cdot\|_{V_{\mp,\kappa}}$, if $\|(\mathrm{Id} - P^\delta){B_\kappa'}^{-1}\mathbb{u}^\delta\|_{V_{\mp,\kappa}} \leq \varepsilon \|\mathbb{u}^\delta\|_U$, then $\gamma^\delta = 1 - \varepsilon$.

Let U^δ be space of piecewise pols of some fixed order w.r.t. quasi-uniform $\mathcal{T}^\delta$ with $h = h_\delta$. Should benefit from $U^\delta \subsetneq U$.

Set $U_1 := H^1(\Omega) \times H_0^1(\Omega)^d$ equipped with sq. norm $(\|f_1\|^2_{L_2(\Omega)} + h^2|f_1|^2_{H^1(\Omega)}) + (\|\vec{f}_2\|^2_{L_2(\Omega)^d} + h^2|\vec{f}_2|^2_{H^1(\Omega)^d})$.
Suppose $\forall \varepsilon > 0$, $\exists V_\mp^\delta = V_\mp^\delta(\varepsilon)$ (with $\dim V_\mp^\delta / \dim U^\delta$ (nearly) indep. of κ) with

$$\|(\mathrm{Id} - P^\delta){B_\kappa'}^{-1}\|_{\mathcal{L}(U_1, V_{\mp,\kappa})} \leq \varepsilon. \tag{9}$$

Then $\|(\mathrm{Id} - P^\delta){B_\kappa'}^{-1}\|_{\mathcal{L}([U,U_1]_{s,2}, V_{\mp,\kappa})} \leq \varepsilon^s$.

For $s \in [0, \frac{1}{2})$, $U^\delta \subset [U, U_1]_{s,2}$, and on U^δ, $\|\cdot\|_{[U,U_1]_{s,2}} \leq C\|\cdot\|_U$. So (uniform) inf-sup stability when $C\varepsilon^s < 1$,

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(9) means $\forall(f_1, \vec{f}_2) \in U_1$, and $(\eta, \vec{v}) \in V_\mp$ with $(\frac{1}{\kappa}\nabla\cdot\vec{v}+\eta, \frac{1}{\kappa}\nabla\eta-\vec{v}) = (f_1, \vec{f}_2)$, $\exists(\eta^\delta, \vec{v}^\delta) \in V_\mp^\delta$ with

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Suff. (very crude)

$$\tfrac{1}{\kappa}\|\nabla\cdot(\vec{v}-\vec{v}^\delta)\| + \|\vec{v}-\vec{v}^\delta\| + \tfrac{1}{\kappa}\|\nabla(\eta-\eta^\delta)\| + \|\eta-\eta^\delta\| \leq \varepsilon\|(f_1, \vec{f}_2)\|_{U_1}. \quad (10)$$

To show quasi-opt. standard Galerkin fem for $\partial\Omega = \Gamma_I$, [Melenk and Sauter, 2011] showed for adjoint problem with hom. bdr. cond. (Schatz) of finding $\phi \in H^1(\Omega)$ s.t. $\int_\Omega \nabla\phi\cdot\nabla\overline{\eta} - \kappa^2\phi\overline{\eta}\,dx \pm i\kappa\int_{\Gamma_I}\phi\overline{\eta}\,ds = \kappa^2\int_\Omega f\overline{\eta}\,dx$ $(\eta \in H^1(\Omega))$, that $\exists\eta^\delta$ from hp-fem space with $\frac{\kappa h}{p}$ and $\frac{\log\kappa}{p}$ small enough, with

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Summary

- ‘practical’ MINRES for an ultra-weak first order system formulation of Helmholtz with optimal test norm is observed to give quasi-best approximations (no polution) when at test side the polynomial degree is moderately larger than that at trial side.
- Hope exists to be able to prove that.
- ‘booster’ version uses the higher order degree at test side to improve the approximation.

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