DPG time-marching scheme with DPG semidiscretization in space for transient advection-reaction equations

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Motivation

Three approaches to discretize **time-dependent PDEs**

Space-time discretizations

From a space-time variational formulation

- Space-time DPG/LS methods

Method of discretization in time

FD in time + sequence of variational problems in space

- BE in time + DPG/LS in space

Method of Lines

Discretization in space + system of ODEs

- Bubnov-Galerkin in space + Exponential Integrators in time

Motivation

Main Focus: **Discontinuous Petrov-Galerkin (DPG)** method

In the context of Method of Lines

- Create a new time-marching scheme based on the DPG method
- Couple it with DPG semidiscretizations in space

Contributions:

[1] J. Muñoz-Matute, D. Pardo, and L. Demkowicz,
Equivalence between the DPG method and the Exponential Integrators for linear parabolic problems,
Journal of Computational Physics, 2021, vol. 429, pp. 110016.

[2] J. Muñoz-Matute, D. Pardo, and L. Demkowicz,
A DPG-based time-marching scheme for linear hyperbolic problems,
Computer Methods in Applied Mechanics and Engineering, 2021, vol. 373, pp. 113539.

[3] J. Muñoz-Matute, L. Demkowicz, and D. Pardo,
Error representation of the time-marching DPG scheme,
Computer Methods in Applied Mechanics and Engineering, 2022, vol. 391, pp. 114480.

[4] J. Muñoz-Matute, L. Demkowicz, and N. Roberts,
Combining DPG in space with DPG time-marching scheme for transient advection-reaction problems,
Computer Methods in Applied Mechanics and Engineering, 2022, In press.

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Model problem and discretization in time

System of ODEs

Let $I = (0, T)$ with $T > 0$

$$\begin{cases} U'(t) + AU(t) = F(t), \ \forall t \in \bar{I} \\ U(0) = U_0 \end{cases}$$

where $A \in \mathbb{R}^{s\times s}$ and $U_0 \in \mathbb{R}^{s\times 1}$.

We define a mesh I_τ of I

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1,$$

where $I_k = (t_{k-1}, t_k)$, $\tau = t_k - t_{k-1}$, $\forall\ k = 1, \ldots, m$ and Γ_τ denotes the mesh skeleton in time.

Variational formulation

Broken ultraweak variational formulation

Denoting by $\mathbb{U} := L^2(I, \mathbb{R}^s)$ and $\mathbb{V} := H^1(I_\tau, \mathbb{R}^s)$

$$\begin{cases} \quad \text{Find } U \in \mathbb{U} \text{ and } \hat{U} = (\hat{U}^1, \ldots, \hat{U}^m) \in \mathbb{R}^{s\times m} \text{ s.t.} \\ B_\tau(U,V) + \langle \hat{U}, V\rangle_{\Gamma_\tau} = \int_I (F(t), V)dt + U_0 V(0), \quad \forall V \in \mathbb{V} \end{cases}$$

where

$$B_\tau(U,V) + \langle \hat{U}, V\rangle_{\Gamma_\tau} := \sum_{k=1}^{m} \int_{I_k} (U, -V' + A^T V)dt - (\hat{U}^k, [V]_k).$$

Here, $(\cdot,\cdot)$ denotes the usual dot product in $\mathbb{R}^s$ and $[V]_k = V(t_k^+) - V(t_k^-), \ \forall k = 1, \ldots, m-1$ and $[V]_m = -V(t_m^-)$.

Optimal test functions

Optimal test functions corresponding to fields

$$\begin{cases} \text{Given } \delta U \in \mathbb{U} \text{ find } V_{\delta U} \in \mathbb{V} \text{ s.t} \\ (V_{\delta U}, \delta V)_{\mathbb{V}} = B_{\tau}(\delta U, \delta V), \ \forall \delta V \in \mathbb{V} \end{cases}$$

Optimal test functions corresponding to traces

$$\begin{cases} \text{Given } \delta \hat{U} \in \mathbb{R}^{s \times m} \text{ find } V_{\delta \hat{U}} \in \mathbb{V} \text{ s.t} \\ (V_{\delta \hat{U}}, \delta V)_{\mathbb{V}} = \langle \delta \hat{U}, \delta V \rangle_{\Gamma_h}, \ \forall \delta V \in \mathbb{V} \end{cases}$$

where the inner product is

$$||V||_{\mathbb{V}}^2 = \sum_{k=1}^{m} || - V' + A^T V||_{I_k}^2 + |V(t_k^-)|^2,$$

where $|| \cdot ||_{I_k}$ denotes the L^2-norm over each time interval I_k.

Optimal test functions for piecewise polynomials

Approximation of the field variables

$$U(t)_{|_{I_k}} \approx U_\tau^k(t) := \sum_{l=0}^{q} U_{\tau,l}^k \left(\frac{t - t_{k-1}}{\tau} \right)^l, \ \forall k = 1, \ldots, m,$$

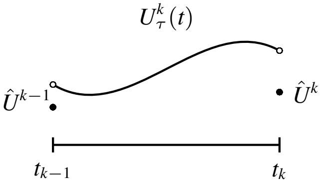
The optimal test functions are defined recursively at each time interval $\forall k = 1, \ldots, m$ as

$$\begin{cases} \hat{V}^k(A^T, t) = e^{A^T(t-t_k)} \\ V_r^k(A^T, t) = \left(A^T\right)^{-1} \left(\left(\frac{t - t_{k-1}}{\tau} \right)^r I_s + \frac{r}{\tau} V_{r-1}^k(A^T, t) - \hat{V}^k(A^T, t) \right) \\ \qquad \forall r = 0, \ldots, q \end{cases}$$

DPG time-marching scheme

$$
\begin{cases}
\hat{U}^k = \hat{V}(A, t_{k-1})\hat{U}^{k-1} + \displaystyle\int_{I_k} \hat{V}(A,t)F(t)dt \\
\displaystyle\sum_{l=0}^{q} U^k_{\tau,l} \int_{I_k} \left(\frac{t - t_{k-1}}{\tau}\right)^{l+r} dt = V^k_r(A, t_{k-1})\hat{U}^{k-1} + \int_{I_k} V^k_r(A,t)F(t)dt
\end{cases}
$$

$\forall r = 0, \ldots, q$ and $\hat{U}^0 = U_0$.



DPG time-marching scheme

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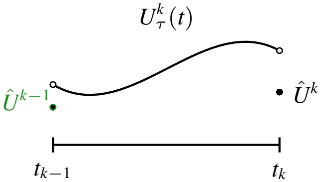

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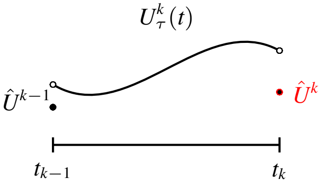

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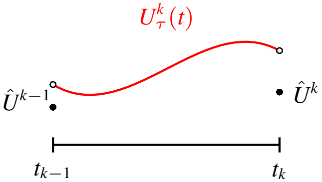


Example: 1D+time heat equation

1D+time heat equation

Let $I = (0, T)$ with $T > 0$ and $\Omega = (0, 1)$

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t), & \text{in } \Omega \times I \\ u(0,t) = u(1,t) = 0, & \text{in } I \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases}$$

<u>Bubnov-Galerkin method in space</u>

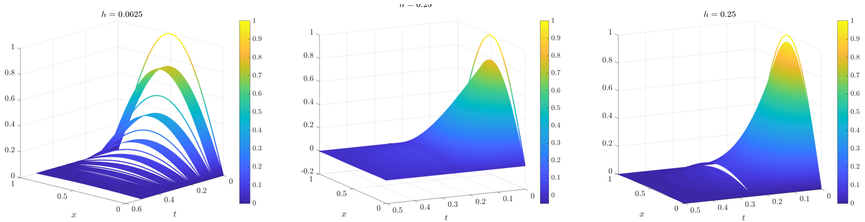
$$\begin{cases} (u_t(t), v) + (u_x(t), v_x) = (f(t), v), & \forall v \in H_0^1(\Omega) \\ (u(0), w) = (u_0, w), & \forall w \in L^2(\Omega) \end{cases}$$

<u>Approximation</u>: $\mathcal{V}_h = \text{span}\{\phi_j(x)\} \subset H_0^1(\Omega)$, $u_h(x,t) = \sum_{j=1}^{s} u_{h,j}(t)\phi_j(x)$

<u>Square system of ODEs</u>: $A = M^{-1}K$

$$\begin{cases} MU'(t) + KU(t) = F(t), \ \forall t \in I \\ M_0 U(0) = U_0 \end{cases}$$

Example: 1D+time heat equation



Approximated solution for $q = 0, 1, 2$.

Model problem and semidiscretization in space

1D+time advection-reaction equation

Let $I = (0, T)$ with $T > 0$ and $\Omega = (0, 1)$

$$\begin{cases} u_t(x,t) + bu_x(x,t) + cu(x,t) = f(x,t), & \text{in } \Omega \times I \\ u(0,t) = g(t), & \text{in } I \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases}$$

where b and c are positive constants.

We define a mesh Ω_h of Ω

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1,$$

where $\Omega_i = (x_{i-1}, x_i)$, $h = x_i - x_{i-1}$, $\forall\, i = 1, \ldots, n$ and Γ_h denotes the mesh skeleton in space.

The logic behind the construction

- Petrov-Galerkin method
- Discontinuous Petrov-Galerkin method
- Practical Discontinuous Petrov-Galerkin method

Petrov-Galerkin formulation

Test space: $\quad H^1_+(\Omega) := \{v \in H^1(\Omega) \mid v(1) = 0\}.$

PG formulation

$$\begin{cases} \text{Find } u \in C^1(\bar{I}; L^2(\Omega)) \text{ s.t. } \forall t \in \bar{I} \\ \quad (u_t(t), v) + \mathfrak{b}(u(t), v) = \tilde{f}(t, v), \qquad \forall v \in H^1_+(\Omega) \\ \qquad\qquad\qquad\qquad (u(0), w) = (u_0, w), \qquad \forall w \in L^2(\Omega) \end{cases}$$

where

$$\mathfrak{b}(u(t), v) := (u(t), -bv_x + cv)$$
$$\tilde{f}(t, v) := (f(t), v) + g(t)bv(0)$$

Petrov-Galerkin discretization

Discrete spaces: $s := \dim(\mathcal{U}_h) = \dim(\mathcal{V}_h)$

$$\mathcal{U}_h := \mathcal{P}^p(\Omega_h), \quad \mathcal{V}_h := \mathcal{P}^{p+1}(\Omega_h) \cap H^1_+(\Omega)$$

PG discretization

$$\begin{cases} \text{Find } u_h \in C^1(\bar{I}; \mathcal{U}_h) \text{ s.t. } \forall t \in \bar{I} \\ (u_{h,t}(t), v_h) + \mathfrak{b}(u_h(t), v_h) = \tilde{f}(t, v_h) \qquad \forall v \in \mathcal{V}_h, \\ (u_h(0), w_h) = (u_0, w_h), \quad \forall w_h \in \mathcal{U}_h \end{cases}$$

Square system of ODEs: $A = M^{-1}K$

$$\begin{cases} MU'(t) + KU(t) = \tilde{F}(t), \ \forall t \in \bar{I} \\ M_0 U(0) = \tilde{U}_0 \end{cases}$$

- Optimal choice of spaces for $u' = f$ (symmetric K).
- Rectangular system in higher dimensions.

DPG formulation

Trial and test spaces: $\mathcal{U} := L^2(\Omega)$ and $\mathcal{V} := H^1(\Omega_h)$

DPG formulation

$$\begin{cases} \text{Find } u \in C^1(\bar{I};\mathcal{U}) \text{ and } \bar{u} = (\bar{u}^1, \ldots, \bar{u}^n) \in C(\bar{I}) \text{ s.t. } \forall t \in \bar{I} \\ \qquad (u_t(t), v) + \mathfrak{b}_h(u(t), v) + \langle \bar{u}(t), v\rangle_{\Gamma_h} = \tilde{f}(t, v), \qquad \forall v \in \mathcal{V} \\ \qquad\qquad (u(0), w) = (u_0, w), \qquad \forall w \in \mathcal{U} \end{cases}$$

We define time-dependent interface variables

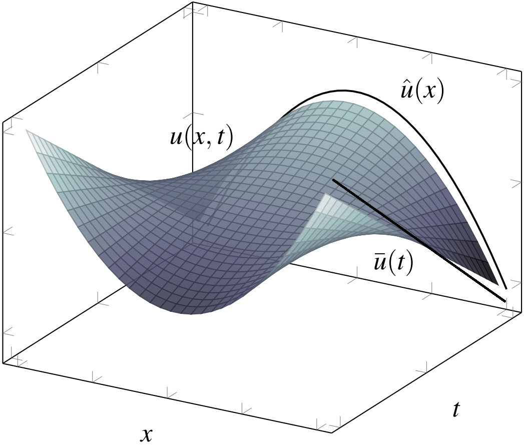
$$\bar{u}^i(t) := u(x_i, t),\ \forall i = 1, \ldots, n,$$

and the bilinear form is

$$\mathfrak{b}_h(u(t), v) + \langle \bar{u}(t), v\rangle_{\Gamma_h} := \sum_{i=1}^{n} (u(t), -bv_x + cv)_{\Omega_i} - \sum_{i=1}^{n} \bar{u}^i(t) b[v]_i,$$

where $[v]_i = v(x_i^+) - v(x_i^-),\ \forall i = 1, \ldots, n-1$ and $[v]_n = -v(x_n^-)$.

DPG discretization



DPG discretization

Selecting

$$\mathcal{U}_h := \mathcal{P}^p(\Omega_h), \quad \mathcal{V}_h := \mathcal{P}^{p+\Delta p}(\Omega_h),$$

with $\Delta p \geq 1$, we obtain a rectangular system

$$\begin{cases} MU'(t) + KU(t) + R\bar{u}(t) = \tilde{F}(t), \ \forall t \in \bar{I} \\ \qquad\qquad\qquad\qquad M_0 U(0) = \tilde{U}_0 \end{cases}$$

Optimal testing

$$\begin{cases} \text{Given } \delta u \in \mathcal{U} \text{ find } v_{\delta u} \in \mathcal{V} \text{ s.t} \\ (v_{\delta u}, \delta v)_{\mathcal{V}} = \mathfrak{b}_h(\delta u, \delta v), \ \forall \delta v \in \mathcal{V}, \end{cases}$$

$$\begin{cases} \text{Given } \delta \bar{u} \in \mathbb{R}^n \text{ find } v_{\delta \bar{u}} \in \mathcal{V} \text{ s.t} \\ (v_{\delta \bar{u}}, \delta v)_{\mathcal{V}} = \langle \delta \bar{u}, \delta v \rangle_{\Gamma_h}, \ \forall \delta v \in \mathcal{V}. \end{cases}$$

Discretization:For any $\delta u_h \in \mathcal{U}_h$ and $\delta \bar{u} \in \mathbb{R}^n$ we have

$$v_{\delta u_h} = G^{-1} K \delta u_h, \quad v_{\delta \bar{u}} = G^{-1} R \delta \bar{u}.$$

DPG discretization

We obtain the following square system of ODEs

$$\begin{cases} K^T G^{-1} M U'(t) + K^T G^{-1} K U(t) + K^T G^{-1} R \bar{u}(t) = K^T G^{-1} \tilde{F}(t), \; \forall t \in \bar{I} \\ R^T G^{-1} M U'(t) + R^T G^{-1} K U(t) + R^T G^{-1} R \bar{u}(t) = R^T G^{-1} \tilde{F}(t) \\ M_0 U(0) = \tilde{U}_0 \end{cases}$$

Eliminating the interface variables

$$\begin{cases} K^T S_1 M U'(t) + K^T S_1 K U(t) = K^T S_1 \tilde{F}(t), \; \forall t \in \bar{I} \\ M_0 U(0) = \tilde{U}_0 \end{cases}$$

where

$$S_1 = G^{-1} - G^{-1} R (R^T G^{-1} R)^{-1} R^T G^{-1}$$

and the matrix A of the system is

$$A = (K^T S_1 M)^{-1} K^T S_1 K.$$

Matrices K, M and G are block diagonal but R is not $\Longrightarrow K^T S_1 M$ is dense
Non practical!

Practical DPG formulation

We maintain the optimal test function $v_{\delta u}$ for the fields and we introduce

$$\begin{cases} \text{Given } \delta\bar{u} \in \mathbb{R}^n \text{ find } v_{\delta\bar{u}} \in \mathcal{V} \text{ and } u \in \mathcal{U} \text{ s.t} \\ (v_{\delta\bar{u}}, \delta v)_{\mathcal{V}} - \mathfrak{b}_h(u, \delta v) = \langle \delta\bar{u}, \delta v\rangle_{\Gamma_h}, \ \forall \delta v \in \mathcal{V} \\ (\delta u, v_{\delta\bar{u}}) = 0, \qquad\qquad \forall \delta u \in \mathcal{U} \end{cases}$$

Discretizing we obtain $v_{\delta\bar{u}} = S_2^T R\delta\bar{u}$ where

$$S_2 = G^{-1} - G^{-1}M(K^TG^{-1}M)^{-1}K^TG^{-1}$$

which leads to the following system of ODEs

$$\begin{cases} K^TG^{-1}MU'(t) + K^TG^{-1}KU(t) + K^TG^{-1}R\bar{u}(t) = K^TG^{-1}\tilde{F}(t), \ \forall t \in \bar{I}, \\ R^TS_2KU(t) + R^TS_2R\bar{u}(t) = R^TS_2\tilde{F}(t), \\ M_0U(0) = \tilde{U}_0, \end{cases} \tag{2.1}$$

Practical DPG formulation

Eliminating the interface variables we obtain the final system

$$\begin{cases} K^T G^{-1} M U'(t) + K^T S_3 K U(t) = K^T S_3 \tilde{F}(t), \ \forall t \in \bar{I} \\ M_0 U(0) = \tilde{U}_0 \end{cases}$$

where

$$S_3 = G^{-1} - G^{-1} R (R^T S_2 R)^{-1} R^T S_2$$

and the matrix is

$$A = (K^T G^{-1} M)^{-1} K^T S_3 K$$

Here, the inversion of matrix $K^T G^{-1} M$ can be computed locally.

The method is consistent with steady-state solutions.

Inner product

Adjoint graph norm

$$||v||_{\mathcal{V}}^2 = \sum_{i=1}^{n} ||v||_{\Omega_i}^2 + || - bv_x + cv||_{\Omega_i}^2$$

Localizable adjoint norm

$$||v||_{\mathcal{V}}^2 = \sum_{i=1}^{n} || - bv_x + cv||_{\Omega_i}^2 + b|v(x_i^-)|^2$$

where $|| \cdot ||_{\Omega_i}$ denotes the L^2-norm over each element Ω_i.

For pure advection ($c = 0$), the adjoint graph norm leads a singular $K^T G^{-1} M$ matrix.

Numerical results

Some observations:

- No difference between employing practical and classical DPG methods and adjoint or adjoint graph norm.
- For pure convection problems the three discretizations in space deliver the same solution.
- For convection-reaction problems, the PG discretization is suboptimal.

<u>Error for the fields</u>:

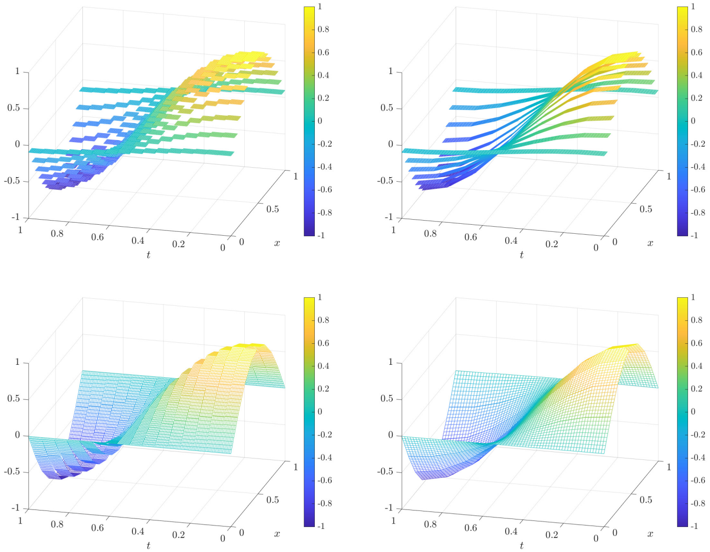
$$\mathcal{E} := \left(\int_I ||u(x,t) - u_{\tau h}(x,t)||^2_{L^2(\Omega)} \, dt \right)^{1/2}$$

<u>Error for the traces</u>:

$$\hat{\mathcal{E}}_1 := \left(\sum_{k=1}^{m} ||u(x,t_k) - \hat{u}_h^k(x)||^2_{L^2(\Omega)} \right)^{1/2}$$

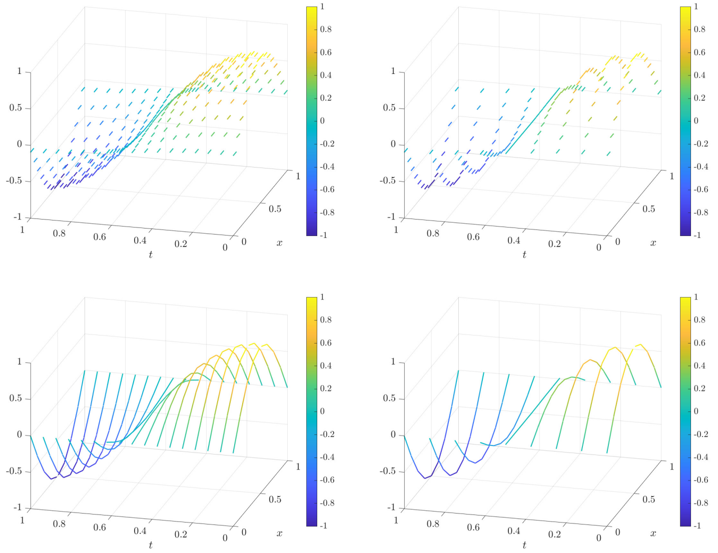
$$\hat{\mathcal{E}}_2 := \max_{1 \leq k \leq m} ||u(x,t_k) - \hat{u}_h^k(x)||_{L^2(\Omega)}$$

Pure advection: smooth solution



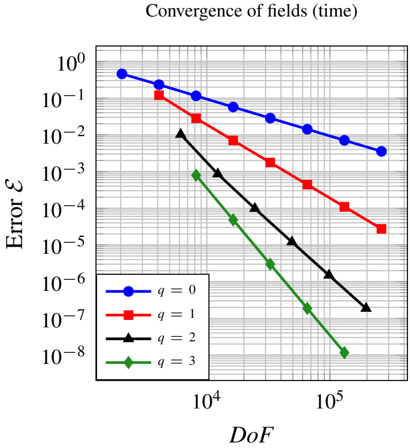
Approximation of fields with $p = 0, 1$ (rows) and $q = 0, 1$ (columns).

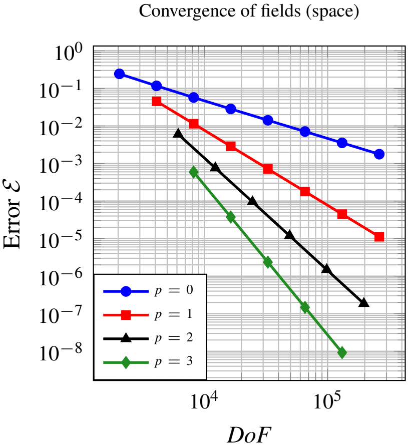
Pure convection: smooth solution



Approximation of traces with $p = 0, 1$ (rows) and $q = 0, 1$ (columns).

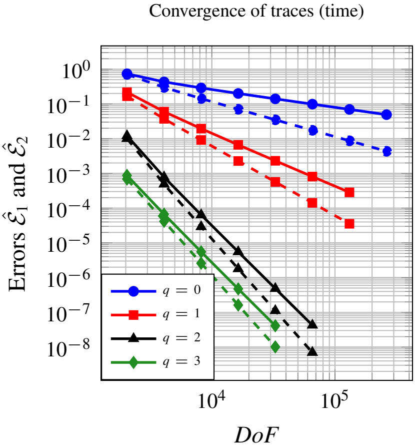
Pure advection: smooth solution

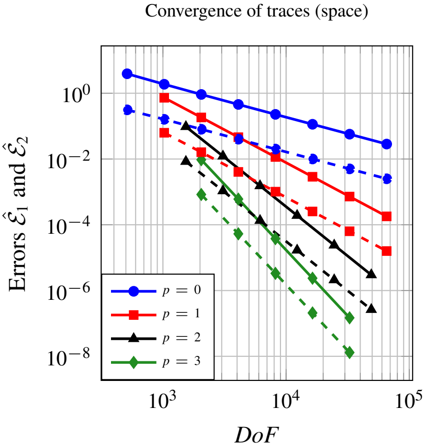




Optimal convergence rates both time and space: $\mathcal{O}(\tau^{q+1})$ and $\mathcal{O}(h^{p+1})$

Pure advection: smooth solution

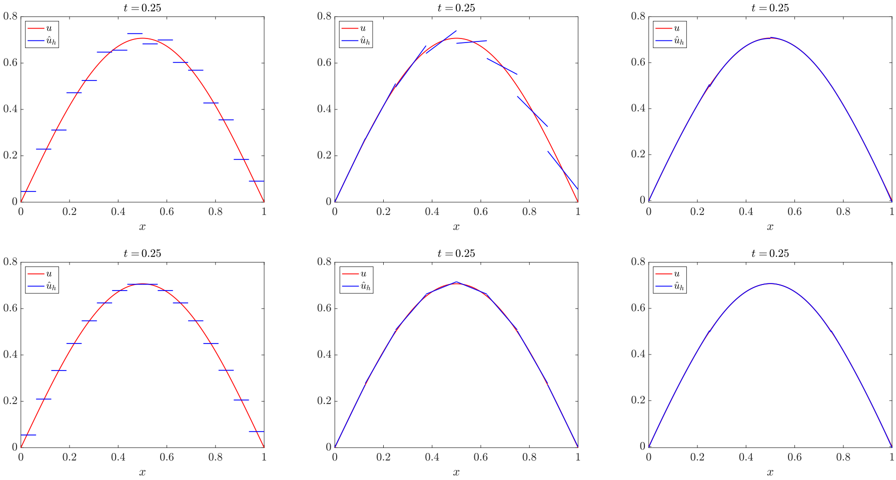




Solid lines refer to error $\hat{\mathcal{E}}_1$ and dashed lines to error $\hat{\mathcal{E}}_2$.

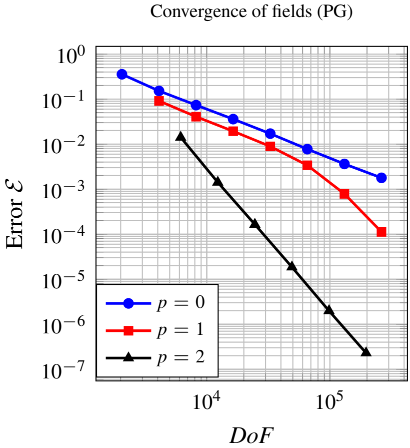
Rates in space for both $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$: $\mathcal{O}(h^{p+1})$
Rate in time for $\hat{\mathcal{E}}_2$: $\mathcal{O}(\tau^{q+1})$ and for $\hat{\mathcal{E}}_1$: $\mathcal{O}(\tau^{q+1/2})$

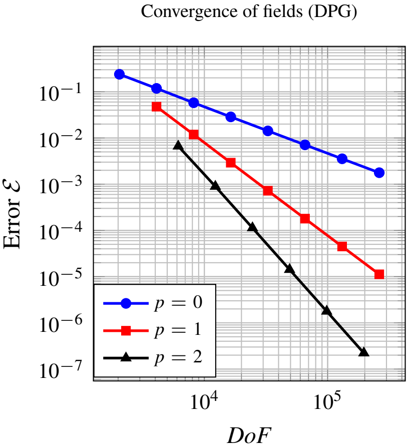
Large reaction term



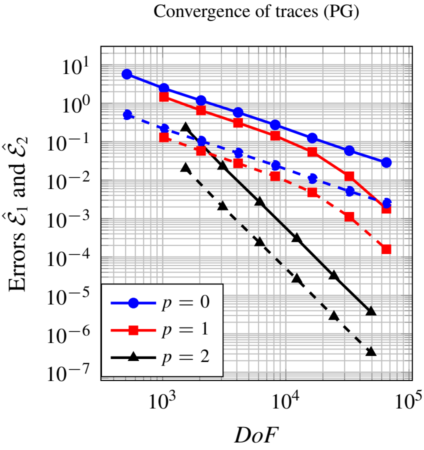
PG method in space (top row) and DPG method (bottom row) for $p = 0, 1, 2$ and $\Delta p = 2$.
Reaction term $c = 10^4$.

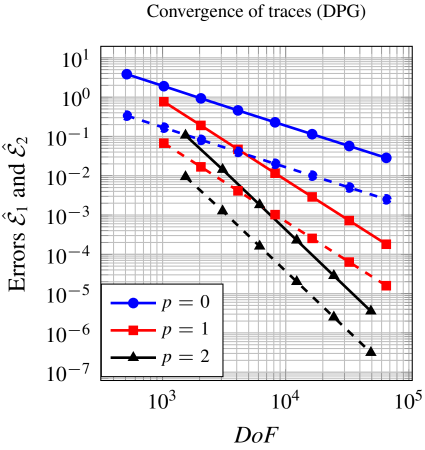
Large reaction term



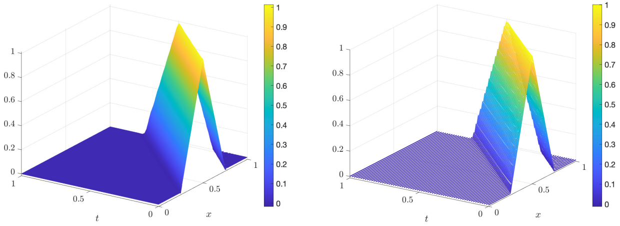


Large reaction term

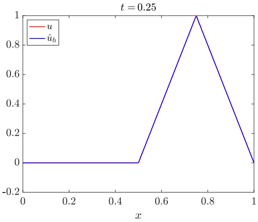




Transport problem: Continuous non-smooth solution

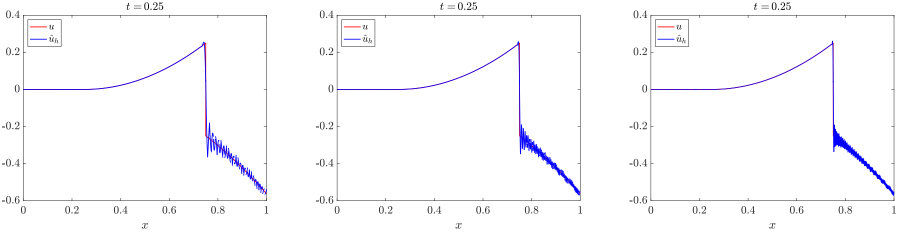


Approximation of fields (left) and traces (center) for $p = q = 1$.



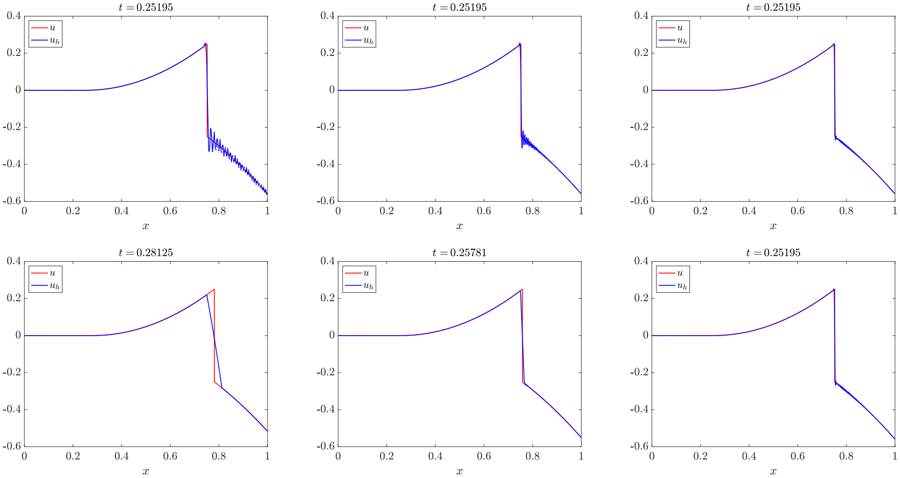
Snapshot at $t = 0.25$.

Transport problem: Discontinuous solution (traces)



Snapshots a discontinuous solution with $q = 0$ and $p = 2$. Trace variables for a fine grid in time and different mesh sizes space.

Transport problem: Discontinuous solution (fields)



Snapshots a discontinuous solution with $q = 0$ and $p = 2$. First row: Field variables for a fine grid in time and different mesh sizes in space. Second row: Field variables for a fine mesh in space and different time step sizes.

Exponential quadrature rule

Time-marching scheme: $\forall k = 1, \ldots, m$ and $\forall r = 0, \ldots, q$

$$\begin{cases} \hat{U}^k = \hat{V}(A, t_{k-1})\hat{U}^{k-1} + \int_{I_k} \hat{V}(A,t)F(t)dt \\ \sum_{l=0}^{q} U_{\tau,l}^k \int_{I_k} \left(\frac{t - t_{k-1}}{\tau}\right)^{l+r} dt = V_r^k(A, t_{k-1})\hat{U}^{k-1} + \int_{I_k} V_r^k(A,t)F(t)dt \end{cases}$$

the optimal test functions can be expressed in terms of the so-called φ-functions

$$\begin{cases} \varphi_0(z) = e^z, \\ \varphi_q(z) = \int_0^1 e^{(1-\theta)z} \frac{\theta^{q-1}}{(q-1)!} d\theta, \ \forall q \geq 1, \end{cases}$$

Lowest order method ($r = 0$)

$$\begin{cases} \hat{U}^k = \hat{U}^{k-1} + \tau\varphi_1(-\tau A)(F(t_{k-1}) - A\hat{U}^{k-1}) \\ U_{\tau,0}^k = \varphi_1(-\tau A)\hat{U}^{k-1} + \tau\varphi_2(-\tau A)F(t_{k-1}) \end{cases}$$

Approximating the action φ-functions

Theorem

Let $A \in \mathbb{R}^{s\times s}$, $W = [w_1, \ldots, w_q] \in \mathbb{R}^{s\times q}$, *and*

$$\tilde{A} = \begin{bmatrix} A & W \\ 0 & J \end{bmatrix} \in \mathbb{R}^{(s+q)\times(s+q)}, \quad J = \begin{bmatrix} 0 & I_{q-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{q\times q};$$

it holds that $\quad e^{\tilde{A}} b_j(1:s) = \sum_{l=1}^{j} \varphi_l(A) w_{j-l+1}, \quad \forall j = 1, \ldots, q,$

b_j *being the j-vector in the canonical basis in* $\mathbb{R}^s$.

Scaling and squaring algorithm + truncated Taylor series

$$e^{\tilde{A}} b_j \approx \left(T_m \left(\frac{1}{\sigma} \tilde{A} \right) \right)^{\sigma} b_j,$$

The values of m and σ are selected based on the sequence of norms: $||\tilde{A}^k||_1^{1/k}$.

[1] A. H. Al-Mohy and N. J. Higham
Computing the action of the matrix exponential, with an application to exponential integrators,
SIAM Journal on Scientific Computing, 33(2):488-511, 2011.

Implementation for DPG semidiscretization

We have $s := \dim(\mathcal{U}_h)$ and $r := \dim(\mathcal{V}_h)$, where $r \gg s$

$$M, K \in \mathbb{R}^{r\times s}, \quad G \in \mathbb{R}^{r\times r}, \quad R \in \mathbb{R}^{r\times n},$$

n being the number of interface variables in space.

Then, we can rewrite the final matrix of the system as

$$A = \underbrace{(K^T G^{-1} M)^{-1} K^T G^{-1} K}_{A_1} - \underbrace{(K^T G^{-1} M)^{-1} K^T G^{-1} R}_{A_2} \underbrace{(R^T S_2 R)^{-1}}_{A_3} \underbrace{R^T S_2 K}_{A_4},$$

where

- $A_1 \in \mathbb{R}^{s\times s}$ and $S_2 \in \mathbb{R}^{r\times r}$ are block diagonal.
- $A_2 \in \mathbb{R}^{s\times n}$ and $A_4 \in \mathbb{R}^{n\times s}$ are sparse and thin.
- $A_3 \in \mathbb{R}^{n\times n}$ is dense but small.

Current work

- Speed-up the computation of φ-functions for matrices with Kronecker sum structure.

J. Muñoz-Matute, D. Pardo, and V. M. Calo,
Exploiting the Kronecker product structure of φ-functions with applications to exponential time integrators,
International Journal for Numerical Methods in Engineering, 2022, pp.1-20.

M. Croci and J. Muñoz-Matute,
Exploiting Kronecker structure in exponential integrators: fast approximation of the action of φ-functions via quadrature,
To be submitted soon.

- Multistage version of the DPG scheme for semi-linear problems.

J. Muñoz-Matute, D. Pardo, and L. Demkowicz,
Multistage DPG time-marching scheme for semi-linear problems,
In preparation.

- MinRes methods in the dual norm employing neural networks.

C. Uriarte, D. Pardo, I. Muga, and J. Muñoz-Matute,
A deep double Ritz method for solving partial differential equations,
To be submitted soon.

Future work

Possible research lines:

- To extend the implementation of the presented DPG semidiscretization in space together with DPG time-marching scheme to higher dimensions.

- (Goal-oriented) adaptivity in space together with adaptivity in time.

- Apply the Kronecker algorithm for 3D+time phase-field models with IGA (GPUs/parallelization).

- Develop an adaptive strategy in time for the Multistage DPG method.

- To employ deep neural networks to approximate the action of the exponential-related functions.