Regularization of rough linear functionals & Adaptivity

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UNITED KINGDOM · CHINA · MALAYSIA

October 5-7, 2022, @MINRES/LS-5, Santiago, Chile

Outline

Motivation (PDEs with rough input data)

Projection in dual norms

Numerical results

Motivation

(PDEs with rough input data)

Motivation

Consider your favorite well-posed variational formulation of a PDE ...

$$\begin{cases} \text{Find } u \in \mathbb{U} \text{ such that:} \\ b(u,v) = \ell(v), \qquad \forall v \in \mathbb{V}. \end{cases}$$

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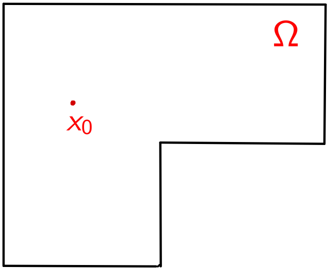

$$\ell(v) = \langle \delta_{x_0}, v \rangle = v(x_0)$$

$$\forall v \in \mathbb{V}.$$

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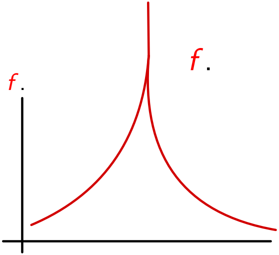
Other examples of rough linear functionals

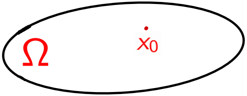
- Singular functions: $\ell(v) = \int_\Omega f\, v$ .... with a singular f.



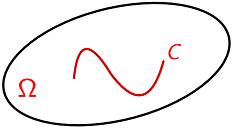
- Action over the derivatives: $\ell(v) = \int_\Omega \vec{F} \cdot \nabla v$.

  ($\vec{F}$ could be singular as well!)

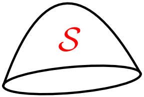
- Point sources: $\ell(v) = \langle \delta_{x_0}, v \rangle = v(x_0)$.



- Line sources: $\ell(v) = \int_C \varphi v$ ..... where C is a contour.



- Surface sources: $\ell(v) = \int_{\mathcal{S}} \psi v$ ..... where $\mathcal{S}$ is a surface.



Regularization idea:

- Find a regularized RHS $\ell_H \approx \ell$ such that $\|\ell - \ell_H\|_{\mathbb{V}^*}$ is controlled.

- Using ℓ_H we define the regularized problem:

$$\begin{cases} \text{Find } u_H \in \mathbb{U} \text{ such that:} \\ b(u_H, v) = \ell_H(v), \qquad \forall v \in \mathbb{V}. \end{cases}$$

- The regularized problem can be approached numerically using AFEM:

$$\begin{cases} \text{Find } u_h \in \mathbb{U}_h \text{ such that:} \\ b(u_h, v_h) = \ell_H(v_h), \qquad \forall v_h \in \mathbb{V}_h. \end{cases}$$

- Error estimation:

$$\|u - u_h\|_{\mathbb{U}} \leq \underbrace{\|u - u_H\|_{\mathbb{U}}}_{\text{regularization}} + \underbrace{\|u_H - u_h\|_{\mathbb{U}}}_{\text{discretization}} \qquad \text{(by triangular inequality)}$$

$$\leq \frac{1}{\gamma} \underbrace{\|\ell - \ell_H\|_{\mathbb{V}^*}}_{\text{regularization}} + \underbrace{\|u_H - u_h\|_{\mathbb{U}}}_{\text{discretization}} \qquad \text{(by inf-sup stability)}$$

2-step adaptive algorithm:

- Set initial mesh $\mathcal{T}_0$ and tolerance $\varepsilon > 0$.
- $[\mathcal{T}_H, \ell_H] = \mathrm{RHS}(\mathcal{T}_0, \ell, \gamma\varepsilon/2)$
- $[\mathcal{T}_h, u_h] = \mathrm{PDE}(\mathcal{T}_H, \ell_H, \varepsilon/2)$

$$\|u - u_h\|_{\mathbb{U}} \leq \frac{1}{\gamma} \underbrace{\|\ell - \ell_H\|_{\mathbb{V}^*}}_{\text{regularization}} + \underbrace{\|u_H - u_h\|_{\mathbb{U}}}_{\text{discretization}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Desirable features for constructing the regularized RHS ℓ_H:

- Must be adaptively built.
- If so, we need to localize the error $\|\ell - \ell_H\|_{\mathbb{V}^*}$ with some indicator.
- We want ℓ_H to be a standard function in some piece-wise polynomial space such that:

$$\langle \ell_H, v_h \rangle = \int_\Omega \ell_H v_h = \sum_j w_j \, \ell_H(x_j) v_h(x_j)$$

Projection in dual norms

Projection is a minimization problem. Let V_H a piecewise polynomial space defined over an affine simplicial mesh $\mathcal{T}_H$. Given a rough $\ell \in \mathbb{V}^*$, we aim to find $\ell_H \in V_H$ such that:

$$\ell_H = \arg\min_{g_H \in V_H} \|\ell - g_H\|_{\mathbb{V}^*} \tag{Min}$$

where

$$\|\cdot\|_{\mathbb{V}^*} := \sup_{v \in \mathbb{V}} \frac{\langle \cdot, v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}}$$

Fortunately, (Min) is equivalent to the mixed (saddle point) formulation:

$$\begin{cases} \text{Find } (r, \ell_H) \in \mathbb{V} \times V_H \quad \text{such that:} \\ \langle J_{\mathbb{V}}(r), v\rangle_{\mathbb{V}^*,\mathbb{V}} + \displaystyle\int_\Omega \ell_H\, v = \ell(v) \qquad \forall v \in \mathbb{V} \\ \displaystyle\int_\Omega g_H\, r = 0 \qquad \forall g_H \in V_H \end{cases}$$

In particular, $\|r\|_{\mathbb{V}}^{q-1} = \|J_{\mathbb{V}}(r)\|_{\mathbb{V}^*} = \|\ell - \ell_H\|_{\mathbb{V}^*}$ (loc. error representative).

Discrete dual norm projection

Recall the mixed (saddle point) formulation:

$$\begin{cases} \text{Find } (r, \ell_H) \in \mathbb{V} \times V_H \quad \text{such that:} \\ \langle J_{\mathbb{V}}(r), v \rangle_{\mathbb{V}^*,\mathbb{V}} + \int_\Omega \ell_H \, v = \ell(v) \qquad \forall v \in \mathbb{V} \\ \int_\Omega g_H \, r = 0 \qquad \forall g_H \in V_H \end{cases}$$

Given a discrete (conforming) piecewise polynomial space $V_h \subset \mathbb{V}$ defined over an affine simplicial mesh $\mathcal{T}_h$, we propose to solve:

$$\begin{cases} \text{Find } (r_h, \ell_H) \in V_h \times V_H \quad \text{such that:} \\ \langle J_{\mathbb{V}}(r_h), v_h \rangle_{\mathbb{V}^*,\mathbb{V}} + \int_\Omega \ell_H \, v_h = \ell(v_h) \qquad \forall v_h \in V_h \\ \int_\Omega g_H \, r_h = 0 \qquad \forall g_H \in V_H \end{cases}$$

Questions:

1. Is this fully-discrete formulation well-posed and stable?
2. How good is this new approximation ℓ_H?
3. Is this new ℓ_H a minimizer in some sense?
4. Can we use $\|r_h\|_{\mathbb{V}}$ as an error indicator for adaptivity?

Under Fortin compatibility condition: There exists a continuous operator $\Pi : \mathbb{V} \to V_h$ such that:

- $\displaystyle\int_\Omega g_H \, \Pi v = \int_\Omega g_H \, v \qquad \forall v \in V_h, \;\; \forall g_H \in V_H$
- $\|\Pi v\|_{\mathbb{V}} \leq C_\Pi \|v\|_{\mathbb{V}}, \qquad \forall v \in \mathbb{V},$ with mesh-independent $C_\Pi > 0$

Example of compatible pairs V_H / V_h :

- $\mathbb{P}_0/(\mathbb{P}_1+\text{bubbles})$ is a compatible pair discretizing $W^{-1,p}(\Omega)/W_0^{1,q}(\Omega)$.
- $\mathbb{P}_1/\mathbb{P}_2$ is a compatible pair discretizing $W^{-1,p}(\Omega)/W_0^{1,q}(\Omega)$.

F. Millar, I. Muga, S. R. & K.G. Van der Zee. *Projection in negative norms and the regularization of rough linear functionals.* Numerische Mathematik (2022) 150:1087–1121.

Fortin compatibility implies ...

1. Is this discrete formulation well-posed and stable? **YES!**

$$\|r_h\|_{\mathbb{V}} \leq \|\ell\|_{\mathbb{V}^*} \qquad \text{and} \qquad \|\ell_H\|_{\mathbb{V}^*} \lesssim C_\Pi \, \|\ell\|_{\mathbb{V}^*}$$

2. How good is this new approximation? **It's quasi-optimal!**

$$\|\ell - \ell_H\|_{\mathbb{V}^*} \leq (1 + 2C_\Pi) \inf_{g_H \in V_H} \|\ell - g_H\|_{\mathbb{V}^*}$$

3. Is this new ℓ_H a minimizer in some sense?
Indeed, it's the minimizer in the discrete dual norm:

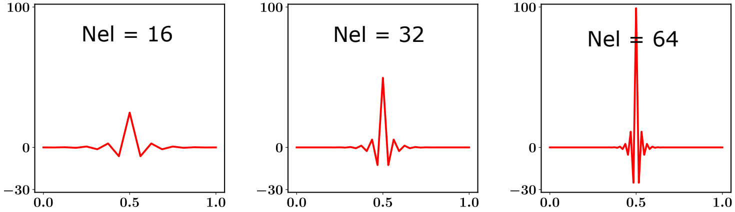
$$\sup_{v_h \in V_h} \frac{\langle \ell - \ell_H, v_h \rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v_h\|_{\mathbb{V}}} \leq \sup_{v_h \in V_h} \frac{\langle \ell - g_H, v_h \rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v_h\|_{\mathbb{V}}} \qquad \forall g_H \in V_H$$

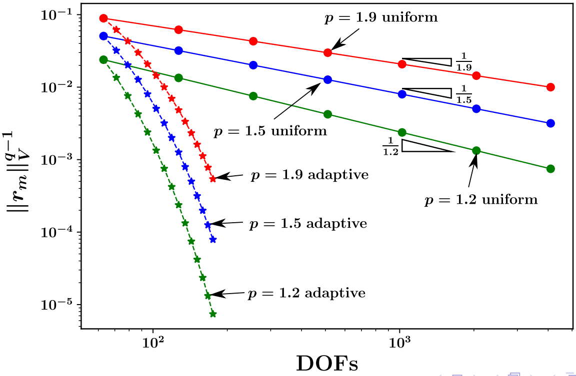
4. Can we use $\|r_h\|_{\mathbb{V}}$ as an error indicator for adaptivity? **YES!**

$$\|r_h\|_{\mathbb{V}}^{q-1} \leq \|\ell - \ell_H\|_{\mathbb{V}^*} \leq C_\Pi \|r_h\|_{\mathbb{V}}^{q-1} + \underbrace{\sup_{v \in \mathbb{V}} \frac{\langle \ell, v - \Pi v \rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}}}_{\operatorname{osc}(\ell)}$$

Numerical Results

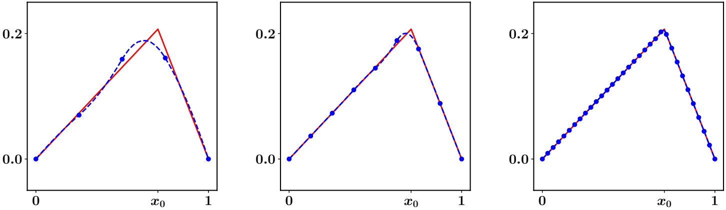
$W^{-1,p}$ projection of Dirac delta in 1D ($\mathbb{P}_1/\mathbb{P}_2$ compatible pair)

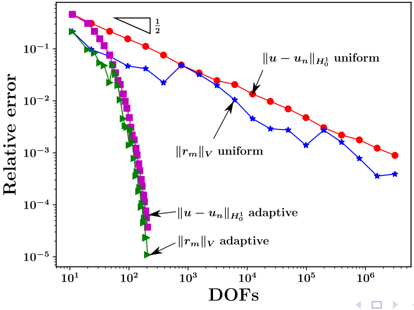




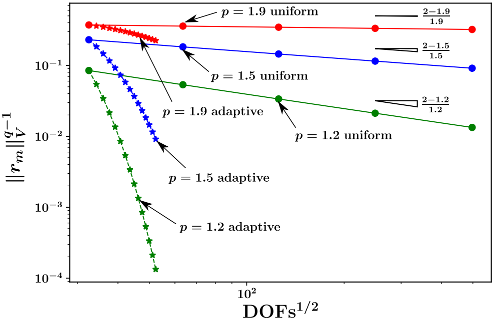
Elliptic ODE with projected Dirac delta source ($p = 2$, $\mathbb{P}_0/\mathbb{P}_2$ C.P.)

$$\begin{cases} -u'' = \delta_{x_0} & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \qquad \text{with } x_0 = \sqrt{2}/2.$$

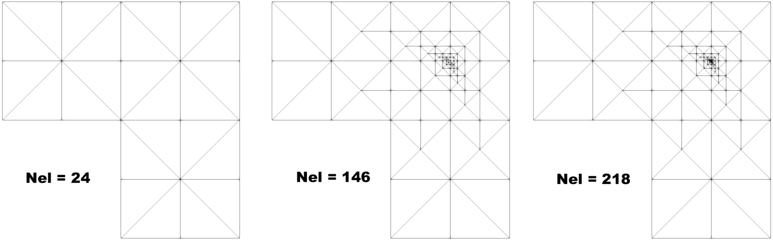




2D Dirac delta projection in $W^{-1,p}$ ($\mathbb{P}_1/\mathbb{P}_2$ C.P.)



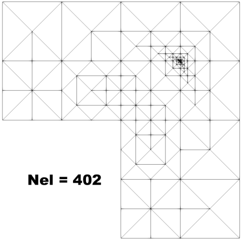
Sequence of mesh refinements for $p = 1.2$

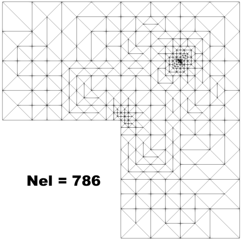


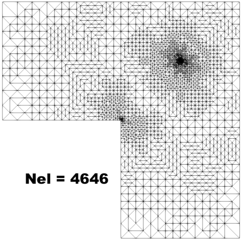
Elliptic PDE on a L-shape domain with Dirac source ($\mathbb{U}_h = \mathbb{P}_1$)

$$\begin{cases} -\Delta u = \delta_{x_0} + f \\ +BCs \end{cases}$$ such that the exact solution is:

$$u(x) = \frac{1}{2\pi} \log\left(\|x - x_0\|\right) + \|x\|^{\frac{2}{3}} \sin\left(\frac{2}{3}(\pi - \theta_x)\right)$$







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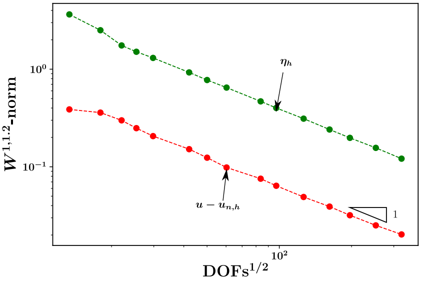
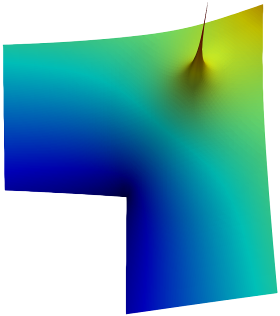
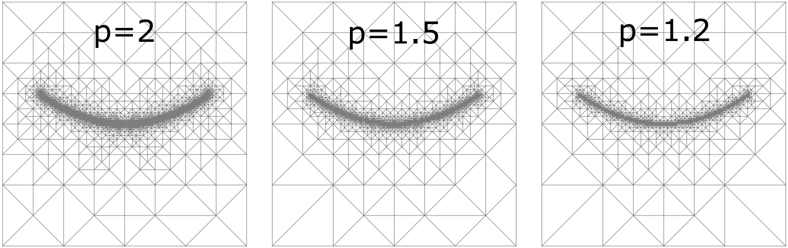
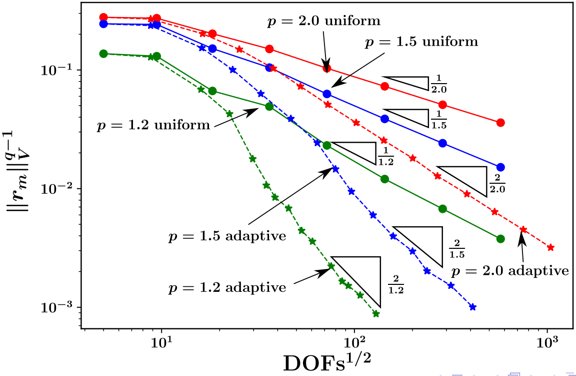




Line source in 2D ($\mathbb{P}_0/\mathbb{P}_1$ + bubbles C.P.)





Future directions

- Rougher RHS (e.g., derivatives of Dirac delta)
- Non-conforming FEM.

Main reference:

F. Millar, I. Muga, S. Rojas & K.G. Van der Zee.
Projection in negative norms and the regularization of rough linear functionals. Numerische Mathematik (2022) 150:1087–1121.

THX !!!

- ANID Fondecyt No 3210009
- European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 777778 (MATHROCKS).

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