A DPG Method for the Quad-div Problem

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joint work with

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Outline

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Model Problem

Find $\boldsymbol{u}$*:*

$$(\nabla \mathrm{div}\,)^2 \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega,$$
$$\mathrm{div}\, \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega.$$

Second-order system

$$\nabla \mathrm{div}\, \boldsymbol{w} + \boldsymbol{u} = \boldsymbol{f}, \quad \nabla \mathrm{div}\, \boldsymbol{u} - \boldsymbol{w} = 0 \quad \text{in} \quad \Omega,$$
$$\mathrm{div}\, \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega.$$

Let $\mathcal{T}$ be a mesh with skeleton $\mathcal{S} = \partial\mathcal{T}$. Testing with $(\boldsymbol{v}, \boldsymbol{\tau})$

$$(\boldsymbol{u}, \boldsymbol{v} - \nabla \mathrm{div}\, \boldsymbol{\tau})_{\mathcal{T}} - (\boldsymbol{w}, \boldsymbol{\tau} + \nabla \mathrm{div}\, \boldsymbol{v})_{\mathcal{T}} + \langle \mathrm{div}\, \boldsymbol{u}, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\mathcal{S}} - \langle \boldsymbol{u} \cdot \boldsymbol{n}, \mathrm{div}\, \boldsymbol{\tau} \rangle_{\mathcal{S}}$$
$$+ \langle \mathrm{div}\, \boldsymbol{w}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\mathcal{S}} - \langle \boldsymbol{w} \cdot \boldsymbol{n}, \mathrm{div}\, \boldsymbol{w} \rangle_{\mathcal{S}} = (\boldsymbol{f}, \boldsymbol{v}).$$

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Outline

Motivation

- Quad-curl problem 2D

$$(\mathbf{curl}\,\mathrm{curl})^2\boldsymbol{u} = ((\nabla\mathrm{div}\,)^2\boldsymbol{u}^\perp)^\perp$$

$\boldsymbol{u} := (u_1, u_2)$ and $\boldsymbol{u}^\perp = (u_2, -u_1)$.

- Linear elasticity (shear-strain energy) [1]

$$(\nabla\mathrm{div}\,\boldsymbol{u})^2.$$

- Magneto-Hydrodynamics [2]
- Electromagnetic inverse scattering theory [3]

[1] R.Mindlin, Arch. Rational Mech. Anal., 16. 1964.
[2] L.Chacón, A.N.Simakov & A. Zocco. Phys. Rev. Lett. 99. 2007.
[3] P.Monk & J.Sun, SIAM J. Sci. Comput., 34(3). 2012.

Outline

DPG method [1]

Find $\mathfrak{u} \in \mathfrak{U}$, *such that*

$$b(\mathfrak{u}, \mathfrak{v}) = L(\mathfrak{v}) \quad \forall \mathfrak{v} \in \mathfrak{V}.$$

$\mathcal{I} : \mathfrak{U} \to \mathfrak{V}$ is the *trial to test* operator

$$\langle\!\langle \mathcal{I}\mathfrak{u} \, , \mathfrak{v} \rangle\!\rangle_{\mathfrak{V}} = b(\mathfrak{u}, \mathfrak{v}) \quad \forall \mathfrak{v} \in \mathfrak{V}.$$

Petrov-Galerkin Scheme: *Find* $\mathfrak{u} \in \mathfrak{U}_h$, *such that*

$$b(\mathfrak{u}_h, \mathfrak{v}) = L(\mathfrak{v}) \quad \forall \mathfrak{v} \in \mathcal{I}(\mathfrak{U}_h)$$

DPG method is *inf-sup stable* and converges *optimally*

$$\|\mathfrak{u} - \mathfrak{u}_h\|_E = \min_{\mathfrak{w} \in \mathfrak{U}_h} \|\mathfrak{u} - \mathfrak{w}\|_E, \quad \text{with} \quad \|\mathfrak{u}\|_E = \|B\mathfrak{u}\|_{\mathfrak{V}'}.$$

[1] L.Demkowicz, J.Gopalakrishnan, Comput. Methods Appl. Mech. Engrg. 2010.

Outline

Ultraweak Formulation

Trial and Test spaces

- $\mathfrak{U} := \mathfrak{U}_0 \times \widehat{\mathfrak{U}}$.
- $\mathfrak{U}_0 := L^2(\Omega) \times L^2(\Omega)$.
- $\widehat{\mathfrak{U}} := (H_0^{1/2}(\mathcal{S}) \times H_0^{-1/2}(\mathcal{S})) \times (H^{1/2}(\mathcal{S}) \times H^{-1/2}(\mathcal{S}))$.
- $\mathrm{H}(\nabla\mathrm{div}, T) = \{\boldsymbol{u} \in L^2(T) : \nabla\mathrm{div}\, \boldsymbol{u} \in L_2(T)\}$ with graph norm $\|\cdot\|^2_{\nabla\mathrm{div}\,,T} := \|\cdot\|^2_T + \|\nabla\mathrm{div}\,(\cdot)\|^2_T$.
- $\mathfrak{V}(\mathcal{T}) := \mathrm{H}(\nabla\mathrm{div}, \mathcal{T}) \times \mathrm{H}(\nabla\mathrm{div}, \mathcal{T})$.

(Bi-)Linear Forms

Let $\mathfrak{u} := (\mathfrak{u}_0, \hat{\mathfrak{u}}) := (\boldsymbol{u}, \boldsymbol{w}, \widehat{\boldsymbol{u}}_{\mathrm{div}}\,, \widehat{\boldsymbol{u}}_n, \widehat{\boldsymbol{w}}_{\mathrm{div}}\,, \widehat{\boldsymbol{w}}_n,) \in \mathfrak{U}$ and $\mathfrak{v} := (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathfrak{V}(\mathcal{T})$

- $b(\mathfrak{u}, \mathfrak{v}) = (\boldsymbol{u}\,, \boldsymbol{v} - \nabla\mathrm{div}\, \boldsymbol{\tau})_{\mathcal{T}} - (\boldsymbol{w}\,, \boldsymbol{\tau} + \nabla\mathrm{div}\, \boldsymbol{v})_{\mathcal{T}} + \langle \widehat{\boldsymbol{u}}_{\mathrm{div}}\,, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\mathcal{S}} - \langle \widehat{\boldsymbol{u}}_n\,, \mathrm{div}\,\boldsymbol{\tau}\rangle_{\mathcal{S}}$ $+\langle \widehat{\boldsymbol{w}}_{\mathrm{div}}\,, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\mathcal{S}} - \langle \widehat{\boldsymbol{w}}_n\,, \mathrm{div}\,\boldsymbol{w}\rangle_{\mathcal{S}}$.
- $L(\mathfrak{v}) = (\boldsymbol{f}\,, \boldsymbol{v})$.

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Ultra-weak formulation

Theorem

Ultra-weak variational formulation has a unique solution $\mathfrak{u} = (\boldsymbol{u}_0, \hat{\boldsymbol{u}}) \in \mathfrak{U}$

$$\|\mathfrak{u}\|_{\mathfrak{U}} \lesssim \|\boldsymbol{f}\|.$$

Sketch of proof

Let $\mathfrak{V}_0 := \{\mathfrak{v} \in \mathfrak{V}(\mathcal{T}) : b(0, \hat{\mathfrak{u}}; \mathfrak{v}) = 0 \quad \forall \hat{\boldsymbol{u}} \in \widehat{\mathfrak{U}}\} = \mathrm{H}_0(\nabla\mathrm{div}, \Omega) \times \mathrm{H}(\nabla\mathrm{div}, \Omega)$, where $\mathrm{H}_0(\nabla\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \mathrm{H}(\nabla\mathrm{div}, \Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = \mathrm{div}\, \boldsymbol{v} = 0 \quad \text{on} \quad \partial\Omega\}$.
If

$$\|\mathfrak{u}_0\|_{\mathfrak{U}_\mathbf{0}} \lesssim \sup_{\mathfrak{v} \in \mathfrak{V}_\mathbf{0} \setminus \{0\}} \frac{|b(\mathfrak{u}_0, 0; \mathfrak{v})|}{\|\mathfrak{v}\|_{\mathfrak{V}}}, \quad \text{and} \quad \|\hat{\mathfrak{u}}\|_{\widehat{\mathfrak{U}}} \lesssim \sup_{\boldsymbol{v} \in \mathfrak{V} \setminus \{0\}} \frac{|b(0, \hat{\mathfrak{u}}; \mathfrak{v})|}{\|\boldsymbol{v}\|_{\mathfrak{V}}},$$

then Inf-Sup condition holds[a]: $\|\mathfrak{u}\|_{\mathfrak{U}} \lesssim \sup_{\mathfrak{v} \in \mathfrak{V}(\mathcal{T}) \setminus \{0\}} \frac{|b(\mathfrak{u}; \mathfrak{v})|}{\|\mathfrak{v}\|_{\mathfrak{V}(\mathcal{T})}}$.

[a] C. Carstensen, L. Demkowicz & J. Gopalakrishnan. Comput. Math. Appl. 72. 2016

Unbroken Adjoint Problem

Problem

For any $(\boldsymbol{f}_1, \boldsymbol{f}_2) \in L_2(\Omega) \times L_2(\Omega)$ find $(\boldsymbol{v}, \boldsymbol{\tau})$ s.t.

$$\boldsymbol{v} - \nabla \mathrm{div}\, \boldsymbol{\tau} = \boldsymbol{f}_1, \quad \boldsymbol{\tau} + \nabla \mathrm{div}\, \boldsymbol{v} = \boldsymbol{f}_2 \quad \text{in} \quad \Omega$$
$$\mathrm{div}\, \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega$$

$$\boldsymbol{v} - \nabla \mathrm{div}\, (\boldsymbol{f}_2 - \nabla \mathrm{div}\, \boldsymbol{v}) = \boldsymbol{f}_1$$

Testing by $\boldsymbol{w} \in \mathrm{H}_0(\nabla \mathrm{div}, \Omega)$ and integration by parts

$$(\nabla \mathrm{div}\, \boldsymbol{v}, \nabla \mathrm{div}\, \boldsymbol{w}) + (\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{f}_1, \boldsymbol{w}) + (\boldsymbol{f}_2, \nabla \mathrm{div}\, \boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathrm{H}_0(\nabla \mathrm{div}, \Omega).$$

The unbroken adjoint problem is well posed!

Outline

Fully Discrete Method[1]

For $\mathfrak{V}_h(\mathcal{T}) \subset \mathfrak{V}(\mathcal{T})$ define and $\mathcal{I}_h : \mathfrak{U}_h \to \mathfrak{V}_h(\mathcal{T})$

$$\langle\!\langle \mathcal{I}_h(\mathfrak{u}) \, , \mathfrak{v} \rangle\!\rangle_{\mathfrak{V}(\mathcal{T})} = b(\mathfrak{u}, \mathfrak{v}) \quad \forall \mathfrak{v} \in \mathfrak{V}_h(\mathcal{T}).$$

Fully discrete method: Find $\mathfrak{u}_h \in \mathfrak{U}_h$

$$b(\mathfrak{u}_h, \mathcal{I}_h(\delta\mathfrak{u})) = L(\mathcal{I}_h(\delta\mathfrak{u})) \quad \forall \delta\mathfrak{u} \in \mathfrak{U}_h.$$

Fortin operator $\Pi_F : \mathfrak{V}(\mathcal{T}) \to \mathfrak{V}_h(\mathcal{T})$

$$b(\mathfrak{u}_h, \mathfrak{v} - \Pi_F \mathfrak{v}) = 0 \quad \forall \mathfrak{u}_h \in \mathfrak{U}_h, \mathfrak{v} \in \mathfrak{V}_h(\mathcal{T}).$$

Discrete stability and quasi-optimal convergence

$$\|\mathfrak{u}_h - \mathfrak{u}\|_{\mathfrak{U}} \lesssim \inf_{\mathfrak{v} \in \mathfrak{V}_h(\mathcal{T})} \|\mathfrak{v} - \mathfrak{u}\|_{\mathfrak{U}}.$$

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Fully Discrete Method

Let T_{ref} the triangle with nodes $(0,0),(1,0),(0,1)$.

Fortin Operator for $\mathrm{H}(\nabla\mathrm{div}, T_{ref})$

Exists $\widetilde{\Pi}_{T_{ref}}^{\nabla\mathrm{div}} : \mathrm{H}(\nabla\mathrm{div}, T_{ref}) \to \mathcal{P}^3(T_{ref})^2$ such that

$$
\begin{aligned}
(\boldsymbol{u}, (1 - \widetilde{\Pi}_{T_{ref}}^{\nabla\mathrm{div}})\boldsymbol{v})_T &= 0 \\
\langle \widehat{\boldsymbol{u}}_n, \mathrm{div}\,(1 - \widetilde{\Pi}_{T_{ref}}^{\nabla\mathrm{div}})\boldsymbol{v}\rangle_{\partial T} &= 0 \\
\langle \widehat{\boldsymbol{u}}_{\mathrm{div}}, (1 - \widetilde{\Pi}_{T_{ref}}^{\nabla\mathrm{div}})\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T} &= 0
\end{aligned}
$$

for all $(\boldsymbol{u}, \widehat{\boldsymbol{u}}_{\mathrm{div}}, \widehat{\boldsymbol{u}}_n) \in \mathcal{P}^0(T_{ref}) \times \mathcal{P}^{1,c}(\partial T_{ref}) \times \mathcal{P}^0(\partial T_{ref})$ and $\boldsymbol{v} \in \mathrm{H}(\nabla\mathrm{div}, T_{ref})$.

Fully Discrete Method

Sketch of Proof

For any $\boldsymbol{v} \in \mathrm{H}(\nabla\mathrm{div}, T_{ref})$ define $\widetilde{\Pi}^{\nabla\mathrm{div}}_{T_{ref}} \boldsymbol{v} := \boldsymbol{v}^*$ solving

$$\begin{aligned}\langle\!\langle \boldsymbol{v}^*, \boldsymbol{\delta v} \rangle\!\rangle_{\nabla\mathrm{div}, T_{ref}} + (\boldsymbol{u}, \boldsymbol{\delta v})_{T_{ref}} + \langle \widehat{\boldsymbol{u}}_{\mathrm{div}}, \boldsymbol{\delta v}\cdot\boldsymbol{n}\rangle_{\partial T_{ref}} + \langle \widehat{\boldsymbol{u}}_n, \mathrm{div}\, \boldsymbol{\delta v}\rangle_{\partial T_{ref}} &= 0,\\ (\boldsymbol{\delta u}, \boldsymbol{v}^*)_{T_{ref}} &= (\boldsymbol{\delta u}, \boldsymbol{v})_{T_{ref}},\\ \langle \delta\widehat{\boldsymbol{u}}_{\mathrm{div}}, \boldsymbol{v}^*\cdot\boldsymbol{n}\rangle_{\partial T_{ref}}, &= (\delta\widehat{\boldsymbol{u}}_{\mathrm{div}}, \boldsymbol{v}\cdot\boldsymbol{n})_{\partial T_{ref}},\\ \langle \delta\widehat{\boldsymbol{u}}_n, \mathrm{div}\, \boldsymbol{v}^*\rangle_{\partial T_{ref}} &= \langle \delta\widehat{\boldsymbol{u}}_n, \mathrm{div}\, \boldsymbol{v}\rangle_{\partial T_{ref}},\end{aligned}$$

for any $(\boldsymbol{\delta u}, \boldsymbol{\delta u}_{\mathrm{div}}, \boldsymbol{\delta u}_n) \in \mathcal{P}^0(T_{ref}) \times \mathcal{P}^{1,c}(\partial T_{ref}) \times \mathcal{P}^0(\partial T_{ref})$.

Rescaling $\widetilde{\Pi}^{\nabla\mathrm{div}}_{T_{ref}}$ for an arbitrary $T \in \mathcal{T}$ is not uniformly bounded with respect to $\mathrm{diam}(T)$!

Fully Discrete Method

Lemma (A. Ern, T. Gudi, I. Smears and M. Vohralík, 2021)

Let $\mathcal{RT}^0(T) = \{c\vec{x} + (a,b)^t : a,b,c \in \mathbb{R},\ \vec{x} \in \mathbb{R}^2\}$. *Exist* $\mathcal{J} : H(\mathrm{div}, T) \to \mathcal{RT}^0(T)$ *with* $\mathrm{div} \circ \mathcal{J} = \Pi^0 \circ \mathrm{div}$ *and*

$$\|\mathcal{J}\boldsymbol{v}\|_T \lesssim \|\boldsymbol{v}\|_T + \mathrm{diam}(T)\|(1-\Pi^0)\mathrm{div}\,\boldsymbol{v}\|_T$$

where $\Pi^0 : L^2(T) \to \mathcal{P}^0(T)$ *is the* $L^2(\Omega)$ *orthogonal projection.*

Lemma (J.Gopalakrishnan and W. Qiu, 2014)

There exists a Fortin operator $\Pi_p^{\mathrm{div}} : H(\mathrm{div}, T) \to P^p(T)^2$

$$(u\,, (1-\Pi_p^{\mathrm{div}})\boldsymbol{v})_T = \langle \hat{u}\,, (1-\Pi_p^{\mathrm{div}})\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T} = 0 \quad \forall \boldsymbol{v} \in H(\mathrm{div}, T)$$

for all $(\boldsymbol{u}, \hat{u}) \in \mathcal{P}^p(T)^2 \times \mathcal{P}^p(\partial T)$ *and* $\mathrm{div} \circ \Pi_{p+1}^{\mathrm{div}} = \Pi^p \circ \mathrm{div}$.

Fully Discrete Method

Proposition

For $T \in \mathcal{T}$ *exists* $\Pi_T^{\nabla\mathrm{div}} : \mathrm{H}(\nabla\mathrm{div}, T) \to \mathcal{P}^3(T)^2$ *such that*

$$\|\Pi_T^{\nabla\mathrm{div}} \boldsymbol{v}\|_{\nabla\mathrm{div}\,,T} \lesssim \|\boldsymbol{v}\|_{\nabla\mathrm{div}\,,T} \quad \forall \boldsymbol{v} \in \mathrm{H}(\nabla\mathrm{div}, T)$$

with

$$b(\boldsymbol{u}, 0, \widehat{\boldsymbol{u}}_{\mathrm{div}}\,, \widehat{\boldsymbol{u}}_n, 0, 0; (1 - \Pi_T^{\nabla\mathrm{div}})\boldsymbol{v}, 0) = 0$$

for all $(\boldsymbol{u}, \widehat{\boldsymbol{u}}_{\mathrm{div}}\,, \widehat{\boldsymbol{u}}_n) \in \mathcal{P}^0(T) \times \mathcal{P}^{1,c}(\partial T) \times \mathcal{P}^0(\partial T)$ *and* $\boldsymbol{v} \in \mathrm{H}(\nabla\mathrm{div}, T)$.

Corollary

$\Pi_F : \mathfrak{V}_h \to \mathfrak{V}$ *with* $\Pi_F \mathfrak{v} = (\Pi_T^{\mathrm{div}} \boldsymbol{v}, \Pi_T^{\nabla\mathrm{div}} \boldsymbol{\tau})$ *for all* $\mathfrak{v} = (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathfrak{V}$, *satisfies the Fortin condition.*

Sketch of proof

- Step 1: Use Piola's tranformation $T_{ref} \to T$, $\widetilde{\Pi}_T^{\nabla\mathrm{div}} : \mathrm{H}(\nabla\mathrm{div}, T) \to \mathcal{P}^3(\mathcal{T})^2$

$$\|\widetilde{\Pi}_T^{\nabla\mathrm{div}} \boldsymbol{v}\|_T \lesssim \|\boldsymbol{v}\| + \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T, \tag{1}$$

$$\|\nabla\mathrm{div}\, \widetilde{\Pi}_T^{\nabla\mathrm{div}} \boldsymbol{v}\|_T \lesssim \mathrm{diam}(T)^{-2} \|\boldsymbol{v}\|_T + \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T. \tag{2}$$

- Step 2: Let $\boldsymbol{v} = \mathcal{J}\boldsymbol{v} + \boldsymbol{w}$ with $\boldsymbol{w} = (1 - \mathcal{J})\boldsymbol{v}$. Helmholtz decomposition $\boldsymbol{w} = \nabla\phi + \mathrm{curl}\psi$ with $\phi = 0$ on ∂T. Also $\Delta\phi = \mathrm{div}\, \boldsymbol{w}$. Then

$$\|\nabla\phi\|_T^2 = -(\Delta\phi, \phi)_T = -(\mathrm{div}\, \boldsymbol{w}, \phi)_T = -((1 - \Pi^0)\mathrm{div}\, \boldsymbol{v}, \phi)_T$$
$$\lesssim \|(1 - \Pi^0)\mathrm{div}\, \boldsymbol{v}\|_T \|\phi\|_T \lesssim \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T \|\nabla\phi\|_T,$$

i.e., $$\|\nabla\phi\|_T \lesssim \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T. \tag{3}$$

- Step 3: Define $\Pi_T^{\nabla\mathrm{div}} \boldsymbol{v} := \mathcal{J} v + \Pi_3^{\mathrm{div}} \mathrm{curl}\Psi + \widetilde{\Pi}_T^{\nabla\mathrm{div}} \nabla\phi$. This operator satisfies the Fortin conditions.
- Step 4: Bound $\Pi_T^{\nabla\mathrm{div}} \boldsymbol{v}$ using (1),(2),(3).

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$$\begin{aligned} \|\nabla\phi\|_T^2 &= -(\Delta\phi, \phi)_T = -(\mathrm{div}\, \boldsymbol{w}, \phi)_T = -((1 - \Pi^0)\mathrm{div}\, \boldsymbol{v}, \phi)_T \\ &\lesssim \|(1 - \Pi^0)\mathrm{div}\, \boldsymbol{v}\|_T \|\phi\|_T \lesssim \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T \|\nabla\phi\|_T, \end{aligned}$$

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Sketch of proof

- Step 1: Use Piola's tranformation $T_{ref} \to T$, $\widetilde{\Pi}_T^{\nabla\mathrm{div}} : \mathrm{H}(\nabla\mathrm{div}, T) \to \mathcal{P}^3(\mathcal{T})^2$

$$\|\widetilde{\Pi}_T^{\nabla\mathrm{div}} \boldsymbol{v}\|_T \lesssim \|\boldsymbol{v}\| + \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T, \quad (1)$$

$$\|\nabla\mathrm{div}\, \widetilde{\Pi}_T^{\nabla\mathrm{div}} \boldsymbol{v}\|_T \lesssim \mathrm{diam}(T)^{-2} \|\boldsymbol{v}\|_T + \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T. \quad (2)$$

- Step 2: Let $\boldsymbol{v} = \mathcal{J}\boldsymbol{v} + \boldsymbol{w}$ with $\boldsymbol{w} = (1 - \mathcal{J})\boldsymbol{v}$. Helmholtz decomposition $\boldsymbol{w} = \nabla\phi + \mathrm{curl}\psi$ with $\phi = 0$ on ∂T. Also $\Delta\phi = \mathrm{div}\, \boldsymbol{w}$. Then

$$\|\nabla\phi\|_T^2 = -(\Delta\phi, \phi)_T = -(\mathrm{div}\, \boldsymbol{w}, \phi)_T = -((1 - \Pi^0)\mathrm{div}\, \boldsymbol{v}, \phi)_T$$
$$\lesssim \|(1 - \Pi^0)\mathrm{div}\, \boldsymbol{v}\|_T \|\phi\|_T \lesssim \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T \|\nabla\phi\|_T,$$

i.e., $$\|\nabla\phi\|_T \lesssim \mathrm{diam}(T)^2 \|\nabla\mathrm{div}\, \boldsymbol{v}\|_T. \quad (3)$$

- Step 3: Define $\Pi_T^{\nabla\mathrm{div}} \boldsymbol{v} := \mathcal{J} v + \Pi_3^{\mathrm{div}} \mathrm{curl}\Psi + \widetilde{\Pi}_T^{\nabla\mathrm{div}} \nabla\phi$. This operator satisfies the Fortin conditions.
- Step 4: Bound $\Pi_T^{\nabla\mathrm{div}} \boldsymbol{v}$ using (1),(2),(3).

Sketch of proof

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Outline

Numerical Experiments

Lowest Order Spaces

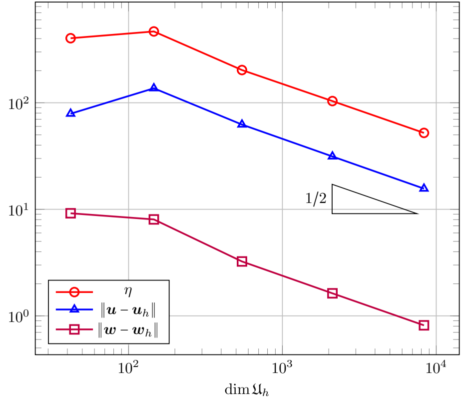
$$\mathfrak{U}_h = \mathfrak{U}_{0,h} \times \widehat{\mathfrak{U}}_h$$
$$\mathfrak{U}_{0,h} = \mathcal{P}^0(\mathcal{T}) \times \mathcal{P}^0(\mathcal{T})$$
$$\widehat{\mathfrak{U}}_h = \mathcal{P}^0(\mathcal{S}) \times \mathcal{P}^{1,c}(\mathcal{S}) \times \mathcal{P}^0(\mathcal{S}) \times \mathcal{P}^{1,c}(\mathcal{S})$$
$$\mathfrak{V}_h(\mathcal{T}) = \mathcal{P}^3(\mathcal{T}) \times \mathcal{P}^3(\mathcal{T})$$

Numerical Experiments

Smooth solution

Let $\Omega = (0,1)^2$

$$\boldsymbol{u} = (x^2(x-1)^2y^2(y-1)^2, \sin(\pi x)^2\sin(\pi y)^2)$$

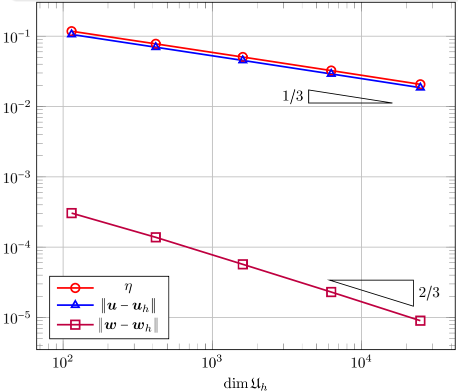


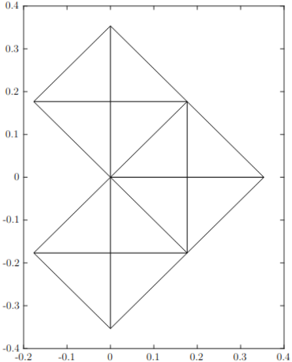
Numerical Experiments

Singular solution

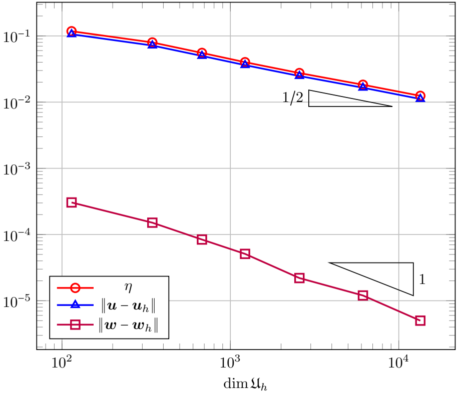
Let $v = r^{2/3}\cos\left(\frac{2}{3}\theta\right)$ in polar coordinates with radius r and angle θ.

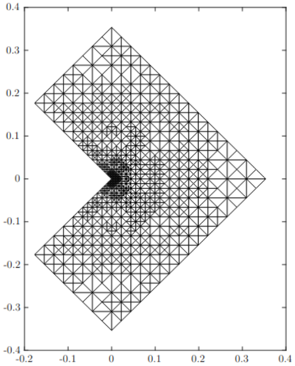
$$\boldsymbol{u} = \mathbf{curl}\, v \in H^{2/3-\varepsilon}(\Omega), \quad \forall \varepsilon > 0,$$
$$\boldsymbol{f} = (\nabla \mathrm{div}\,)^2 \boldsymbol{u} + \boldsymbol{u}.$$





Numerical Experiments





Conclusions

- Second-order reformulation of Quad-Div problem
- Well-posded ultra-weak formulation
- Fully discrete DPG method
  - Construction of Fortin operator
  - Lowest-order discretization

Future works:

- Quad-div in 3D.
- Quad-curl in 3D.

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