MINRES/LS-5, Santiago, Chile

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MINRES for PDEs with singular data

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joint work with

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Introduction & Motivation

Shell problems

Water tower Möglingen

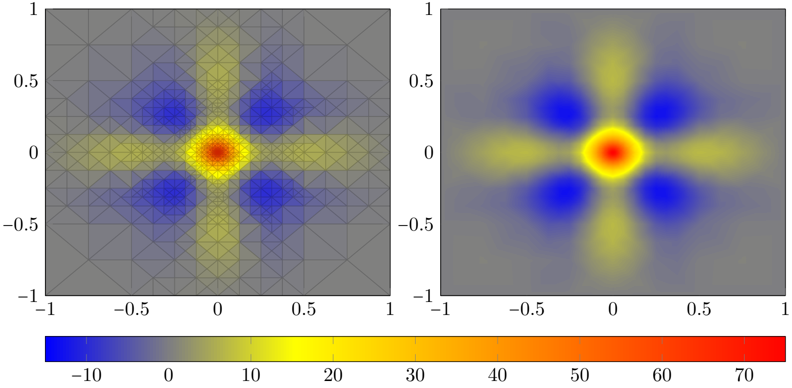
- Hyperbolic shell structure
- Complex stress concentrations occur in shell deformations

wikipedia.org

DPG for Koiter shell (see talk of Antti & Norbert)

Coupled problem (4th–2nd order PDE)

Hyperbolic shell with point load at origin



F., Heuer, Niemi: A DPG method for shallow shells, Numer. Math. 2022

Poisson problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega,\\
u|_{\partial\Omega} &= 0.
\end{aligned}
$$

First-order formulation

$$
\begin{aligned}
-\operatorname{div} \boldsymbol{\sigma} &= f \quad \text{in } \Omega,\\
\nabla u - \boldsymbol{\sigma} &= 0 \quad \text{in } \Omega,\\
u|_{\partial\Omega} &= 0.
\end{aligned}
$$

Weak formulation

Find solution $u \in H^1_0(\Omega)$ of

$$(\nabla u\,, \nabla v) = (f\,, v) \quad \forall v \in H^1_0(\Omega)$$

LS formulation

Find minimizer $(u, \boldsymbol{\sigma})$ of

$$\|\operatorname{div} \boldsymbol{\tau} + f\|^2 + \|\nabla v - \boldsymbol{\tau}\|^2$$

FOSLS

Continuous space

$$W = H_0^1(\Omega) \times \boldsymbol{H}(\mathrm{div}\,;\Omega) = H_0^1(\Omega) \times \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{R}^n) \,:\, \mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)\}$$

Least-squares finite element methods: Bochev & Gunzberger '09

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Discrete space

$$W_h = \mathcal{S}_0^1(\mathcal{T}) \times \mathcal{RT}^0(\mathcal{T}) \subseteq W.$$

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Error: If $f \in L^2(\Omega)$ then

$$\|u - u_h\|_1^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{H}(\mathrm{div}\,;\Omega)}^2 \eqsim \|\mathrm{div}\,\boldsymbol{\sigma}_h + f\|^2 + \|\nabla u_h - \boldsymbol{\sigma}_h\|^2$$

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Asymptotic exactness of FOSLS: Carstensen & Storn, SINUM 2018.

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FOSLS

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Regularized FOSLS

$$(u_h, \boldsymbol{\sigma}_h) = \underset{(v_h, \boldsymbol{\tau}_h) \in W_h}{\arg\min} \|\operatorname{div} \boldsymbol{\tau}_h + Q_h^\star f\|^2 + \|\nabla v_h - \boldsymbol{\tau}_h\|^2$$

- $Q_h^\star \colon H^{-1}(\Omega) \to \mathcal{P}^p(\mathcal{T})$ (quasi-)interpolation/projection
- How do we construct such an operator?

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Compute $Q_h^\star f$ by solving a Petrov–Galerkin problem:

- Millar, Muga, Rojas, van der Zee (Numer. Math. 2022)
- Requires to solve global problem

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- Requires to solve global problem

Our construction of $Q_h^\star f$:

- Local, thus efficient to evaluate
- Bounded in $H^{-1}(\Omega)$ as well as $L^2(\Omega)$
- Idempotent: $Q_h^\star \phi = \phi$ for ϕ pw. constant

An alternative

Discrete H^{-1} norm (Bramble, Lazarov, Pasciak '97)

$$(u_h, \boldsymbol{\sigma}_h) = \underset{(v_h, \boldsymbol{\tau}_h) \in W_h}{\arg\min} \|\operatorname{div} \boldsymbol{\tau}_h + f_h\|_{-1,h}^2 + \|\nabla v_h - \boldsymbol{\tau}_h\|^2$$

- Requires a polynomial approximation f_h of f
- Implementation and analysis of discrete H^{-1} inner product
- Quasi-optimality and optimal error estimate

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim h^s \|f\|_{-1+s}$$

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Our regularized FOSLS

$$(u_h, \boldsymbol{\sigma}_h) = \underset{(v_h, \boldsymbol{\tau}_h) \in W_h}{\arg\min} \|\operatorname{div} \boldsymbol{\tau}_h + Q_h^\star f\|^2 + \|\nabla v_h - \boldsymbol{\tau}_h\|^2$$

- Use code that you have: $Q_h^\star f$ as right-hand side
- We will prove quasi-optimality and

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim h^s \|f\|_{-1+s}$$

Local (quasi-)projection operators in H^{-1}

The lowest-order FOSLS is a regularized FOSLS (with orthogonal proj.)

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L^2 orthogonal projection $\Pi_h^0 \colon L^2(\Omega) \to \mathcal{P}^0(\mathcal{T})$ not bounded in $H^{-1}(\Omega)$

$$\|\Pi_h^0 \phi\|_{-1} \not\lesssim C \|\phi\|_{-1}$$

Though,

$$\|\phi - \Pi_h^0 \phi\|_{-1} \lesssim \|h_{\mathcal{T}}(1 - \Pi_h^0)\phi\|.$$

shows that it is bounded when restricted to polynomials:

$$\|\Pi_h^0 \phi\|_{-1} \lesssim \|\phi\|_{-1} + \|h_{\mathcal{T}} \phi\| \lesssim \|\phi\|_{-1} \quad \forall \phi \in \mathcal{P}^p(\mathcal{T}).$$

Quasi-interpolation in $H^1_0(\Omega)$

$J_h \colon L^2(\Omega) \to \mathcal{P}^1(\mathcal{T}) \cap H^1_0(\Omega)$,

$$J_h v = \sum_{z \in \mathcal{N}_0} (v\,, \varphi_z)\eta_z,$$

$\varphi_z \in \mathcal{P}^1(\omega_{\mathcal{T}}(z))$ dual basis with $(\varphi_z\,, \eta_{z'}) = \delta_{z,z'}$.

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Explicit representation

$$\varphi_z|_{\Omega(z)} = \frac{1}{|\Omega(z)|}\big((n+1)(n+2)\eta_z - (n+1)\big).$$

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$$\varphi_z|_{\Omega(z)} = \frac{1}{|\Omega(z)|}((n+1)(n+2)\eta_z - (n+1)).$$

Bubble operator: scaled bubble function $\eta_{b,T}$, characteristic function χ_T

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Fortin-type operator:

$$P_h = J_h + B_h(1 - J_h)$$

Adjoint operator

Fortin-type operator: constructed so that $(1\,,(1-P_h)v)_T = 0$

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Adjoint operator: $P_h' \colon H^{-1}(\Omega) \to \mathcal{P}^1(\mathcal{T})$

$$P_h'\phi = J_h'\phi + (1 - J_h')B_h'\phi$$

where

$$J_h'\phi = \sum_{z\in\mathcal{N}_0} (\phi\,,\eta_z)\varphi_z, \quad B_h'\phi = \sum_{T\in\mathcal{T}} (\phi\,,\eta_{b,T})\chi_T$$

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It is not a projection but $P_h'\phi = \phi$ if $\phi \in \mathcal{P}^0(\mathcal{T})$

To obtain a projection operator: Apply Π_h^0:

$$Q_h := \Pi_h^0 P_h' \quad \text{then } Q_h^2 = \Pi_h^0 \underbrace{P_h'\Pi_h^0}_{=\Pi_h^0} P_h' = \Pi_h^0 P_h' = Q_h.$$

Theorem (F. '21)

$Q_h^\star \in \{P_h', Q_h\}$ $(\mathrm{ran}(P_h') \subset \mathcal{P}^1(\mathcal{T})$, $\mathrm{ran}(Q_h) = \mathcal{P}^0(\mathcal{T}))$

- Idempotent: $Q_h^\star \phi = \phi$ for all $\phi \in \mathcal{P}^0(\mathcal{T})$
- Approximation:

$$\|(1 - Q_h^\star)\phi\|_{-1} \lesssim \|h_{\mathcal{T}}\phi\|$$

- Global boundedness: (is also true locally)

$$\|Q_h^\star \phi\|_{-1} \lesssim \|\phi\|_{-1}, \quad \|Q_h^\star \phi\| \lesssim \|\phi\|$$

- Efficient computation (if (ϕ, η_z) and $(\phi, \eta_{b,T})$ known)
- Interpolation theory proves corresponding results in $H^{-s}(\Omega)$

Multilevel norms and decompositions in negative order Sobolev spaces, Math. Comp. '21

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Other interpolation/projection operators $\Pi\colon H^{-1}(\Omega) \to L^2(\Omega)$

- Diening, Storn, Tscherpel: arXiv.org, 2021+
- Stevenson & van Venetië, 2018
- Veeser, Zanotti, ...

Multilevel norms and decompositions in negative order Sobolev spaces, Math. Comp. '21

Regularized FOSLS

The standard lowest-order FOSLS

$$(u_h, \boldsymbol{\sigma}_h) = \underset{(v_h, \boldsymbol{\tau}_h) \in W_h}{\arg\min} \|\operatorname{div} \boldsymbol{\tau}_h + \Pi_h^0 f\|^2 + \|\nabla v_h - \boldsymbol{\tau}_h\|^2$$

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Proposition (Cai, Lazarov, Manteuffel, McCormick, '94)

Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ solution of $-\Delta u = f$. Then,

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{H}(\operatorname{div};\Omega)} \lesssim h^{s_\Omega} \|u\|_{1+s_\Omega} + \|(1 - \Pi_h^0) f\|.$$

$1/2 < s_\Omega \leq 1$ is the regularity shift ($s_\Omega = 1$ if Ω convex)

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Estimate in weaker norm

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim h^{s_\Omega} \|\Pi_h^0 f\|_{-1+s_\Omega} + h\|(1 - \Pi_h^0) f\|.$$

Observation: If $f \in \mathcal{P}^0(\mathcal{T})$, then last term vanishes

Proof idea

Estimate in weaker norm

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Note: $(u_h, \boldsymbol{\sigma}_h)$ is approximation of
Regularized problem: $-\Delta \widetilde{u} = \Pi_h^0 f$, $\widetilde{\boldsymbol{\sigma}} = \nabla \widetilde{u}$ with error

$$\|u - \widetilde{u}\|_1 + \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\| \lesssim \|(1 - \Pi_h^0) f\|_{-1} \lesssim h\|(1 - \Pi_h^0) f\|.$$

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Standard FOSLS error estimate:

$$\|\widetilde{u}-u_h\|_1 + \|\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h\|_{\boldsymbol{H}(\mathrm{div}\,;\Omega)} \lesssim h^{s_\Omega}\|\Pi_h^0 f\|_{-1+s_\Omega} + \|(1-\Pi_h^0)\Pi_h^0 f\|.$$

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Theorem (F., Heuer, Karkulik, '22)

Let $f \in H^{-1+s}(\Omega)$ for some $0 \le s \le s_\Omega$. Then,

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim h^s \|f\|_{-1+s}.$$

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Proof idea: (same as before with $\Pi_h^0 f$ replaced by $Q_h f$)
Regularized problem: $-\Delta \widetilde{u} = Q_h f$, $\widetilde{\boldsymbol{\sigma}} = \nabla \widetilde{u}$ with error

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Standard FOSLS error estimate:

$$\|\widetilde{u} - u_h\|_1 + \|\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h\|_{\boldsymbol{H}(\operatorname{div};\Omega)} \lesssim h^s\|Q_h f\|_{-1+s} \lesssim h^s\|f\|_{-1+s}$$

Quasi-optimality

$$(u_h, \boldsymbol{\sigma}_h) = \underset{(v_h, \boldsymbol{\tau}_h) \in W_h}{\arg\min} \|\operatorname{div} \boldsymbol{\tau}_h + Q_h f\|^2 + \|\nabla v_h - \boldsymbol{\tau}_h\|^2$$

Theorem (F. 22+)

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim \min_{(v_h, \boldsymbol{\tau}_h) \in W_h} \|u - v_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|$$

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Proof ingredients $(\boldsymbol{\sigma} = \nabla u)$

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim \min_{v_h \in \mathcal{S}_0^1(\mathcal{T})} \|u - v_h\|_1 + \|(1 - Q_h) f\|_{-1}$$

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$$(u_h, \boldsymbol{\sigma}_h) = \operatorname*{arg\,min}_{(v_h, \boldsymbol{\tau}_h) \in W_h} \|\operatorname{div} \boldsymbol{\tau}_h + Q_h f\|^2 + \|\nabla v_h - \boldsymbol{\tau}_h\|^2$$

Theorem (F. 22+)

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim \min_{(v_h, \boldsymbol{\tau}_h) \in W_h} \|u - v_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|$$

Proof ingredients ($\boldsymbol{\sigma} = \nabla u$)

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim \min_{v_h \in \mathcal{S}_0^1(\mathcal{T})} \|u - v_h\|_1 + \|(1 - Q_h) f\|_{-1}$$

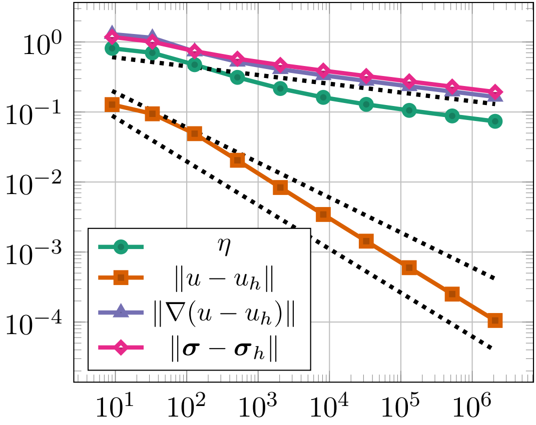
Using that Q_h is a projection on $\mathcal{P}^0(\mathcal{T})$ and $f = -\operatorname{div} \boldsymbol{\sigma}$

$$\begin{aligned}\|(1 - Q_h) f\|_{-1} = \|(1 - Q_h)\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|_{-1} &\lesssim \|\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|_{-1} \\ &\lesssim \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|\end{aligned}$$

for $\boldsymbol{\tau}_h \in \{\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}; \Omega) : \operatorname{div} \boldsymbol{\tau} \in \mathcal{P}^0(\mathcal{T})\}$, particularly $\boldsymbol{\tau}_h \in \mathcal{RT}^0(\mathcal{T})$

Numerical experiment

$$u(x,y) = |x-y|^{3/4}\sin(\pi x)\sin(\pi y) \in H^{1+1/4-\varepsilon}(\Omega), \quad (x,y)\in(0,1)^2,$$
$$f = -\Delta u \in H^{-1+1/4-\varepsilon}(\Omega).$$



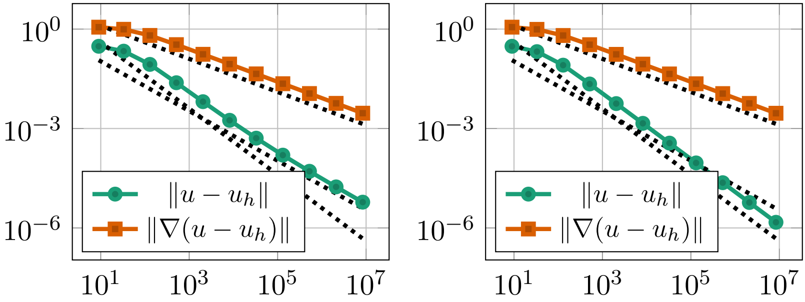
Dotted lines: $\mathcal{O}(h^{1/4})$, $\mathcal{O}(h)$, $\mathcal{O}(h^{1+1/4})$

L^2 estimate in FOSLS

Situation: $f \in L^2(\Omega)$, so no need to regularize data, but

- $\|u - u_h\| = \mathcal{O}(h^2)$ if $Q_h^\star = Q_h$ (regularized FOSLS)
- $\|u - u_h\| = \mathcal{O}(h^{3/2})$ if $Q_h^\star = \Pi_h^0$ (standard FOSLS)

$Q_h^\star = \Pi_h^0$ $\qquad$ $Q_h^\star = Q_h$



You can prove that regularized FOSLS converges optimal in L^2 norm!

Cai, Ku, SINUM '06, '10 and Ku, SINUM '11

A posteriori estimate

$$\eta^2 := \|\nabla u_h - \boldsymbol{\sigma}_h\|^2 + \|\operatorname{div} \boldsymbol{\sigma}_h + Q_h f\|^2,$$
$$\operatorname{osc}(f) := \|(1 - Q_h) f\|_{-1}.$$

Ern, Gudi, Smears, Vohralík, IMA, '22

Carstensen, Collective marking for adaptive LSFEM, Math. Comp. Vol. 89, 2020

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Theorem (F., Heuer, Karkulik, '22)

If $f \in H^{-1}(\Omega), \eta \lesssim \|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim \eta + \operatorname{osc}(f)$

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This requires duality-type argument and

- Interpolation operator in Raviart–Thomas space (Ern et al. '21)
- Similar estimate known if $f \in L^2(\Omega)$ with $\operatorname{osc}(f) = \|(1 - \Pi_h^0) f\|_{-1} \lesssim \|h_{\mathcal{T}}(1 - \Pi_h^0) f\|$

Ern, Gudi, Smears, Vohralík, IMA, '22

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Regularized DPG

Ultra-weak formulation of Poisson problem

- $\boldsymbol{u} = (u, \boldsymbol{\sigma}, \widehat{u}, \widehat{\sigma}) \in U$, $\boldsymbol{v} = (v, \boldsymbol{\tau}) \in V$
- Bilinear form $b : U \times V \to \mathbb{R}$, Functional $L : V \to \mathbb{R}$

$$
\begin{aligned}
b(\boldsymbol{u}, \boldsymbol{v}) &:= (\boldsymbol{\sigma}\,, \nabla_{\mathcal{T}} v) - \langle \widehat{\sigma}\,, v \rangle_{\mathcal{S}} \\
&\qquad + (\boldsymbol{\sigma}\,, \boldsymbol{\tau}) + (u\,, \operatorname{div}_{\mathcal{T}} \boldsymbol{\tau}) - \langle \widehat{u}\,, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\mathcal{S}}, \\
L(\boldsymbol{v}) &:= (f, v)
\end{aligned}
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- Variational form: Find $\boldsymbol{u} \in U$ s.t.

$$b(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V.$$

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Theorem (Demkowicz & Gopalakrishnan, SINUM '12)

$$C^{-1} \|\boldsymbol{u}\|_U \le \|B\boldsymbol{u}\|_{V'} = \sup_{\boldsymbol{v} \in V} \frac{b(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_V} = b(\boldsymbol{u}, \Theta \boldsymbol{u})^{1/2} \quad \forall \boldsymbol{u} \in U$$

Framework

- $\mathcal{T}$ mesh with skeleton $\mathcal{S}$
- Traces:

$$\gamma_{0,\mathcal{S}} : H^1(\Omega) \to \prod_{T \in \mathcal{T}} H^{1/2}(\partial T) \qquad \gamma_{0,\mathcal{S}} v|_{\partial T} = v|_{\partial T},$$

$$\gamma_{\boldsymbol{n},\mathcal{S}} : \boldsymbol{H}(\mathrm{div}\,; \Omega) \to \prod_{T \in \mathcal{T}} H^{-1/2}(\partial T) \quad \gamma_{\boldsymbol{n},\mathcal{S}} \boldsymbol{\sigma}|_{\partial T} = \boldsymbol{\sigma} \cdot \boldsymbol{n}_T$$

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- Spaces: $U := L^2(\Omega) \times L^2(\Omega)^n \times H_0^{1/2}(\mathcal{S}) \times H^{-1/2}(\mathcal{S})$

$$H_0^{1/2}(\mathcal{S}) := \gamma_{0,\mathcal{S}}(H_0^1(\Omega)),$$
$$H^{-1/2}(\mathcal{S}) := \gamma_{\boldsymbol{n},\mathcal{S}}(\boldsymbol{H}(\mathrm{div}\,;\Omega))$$

- Norms (minimum energy extension)

$$\|(\widehat{u},\widehat{\sigma})\|_{\mathcal{S}}^2 := \|\widehat{u}\|_{1/2,\mathcal{S}}^2 + \|\widehat{\sigma}\|_{-1/2,\mathcal{S}}^2$$

Practical DPG

Spaces

- $U_h := \mathcal{P}^0(\mathcal{T}) \times \mathcal{P}^0(\mathcal{T})^n \times \mathcal{P}^1_c(\mathcal{S}) \times \mathcal{P}^0(\mathcal{S})$
- $V_h := \mathcal{P}^n(\mathcal{T}) \times \mathcal{P}^2(\mathcal{T})^n$

Fortin operator (Gopalakrishnan & Qiu, Math. Comp. 2014)

Existence of $\Pi : V \to V_h$ s.t. for all $\boldsymbol{u}_h \in U_h$, $\boldsymbol{v} \in V$

$$\|\Pi \boldsymbol{v}\|_V \leq C_\Pi \|\boldsymbol{v}\|_V, \quad b(\boldsymbol{u}_h, \boldsymbol{v}) = b(\boldsymbol{u}_h, \Pi \boldsymbol{v})$$

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Still: Quasi-best approximation

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_U \leq C \min_{\boldsymbol{w}_h \in U_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_U$$

Regularized DPG method

$$b(\boldsymbol{u}, \boldsymbol{v}) = (f\,, v) \quad \forall \boldsymbol{v} = (v, \boldsymbol{\tau}) \in V = H^1(\mathcal{T}) \times \boldsymbol{H}(\mathrm{div}\,; \mathcal{T}).$$

Again, taking $f \in H^{-1}(\Omega)$ is not well defined

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Theorem (F., Heuer, Karkulik, '22)

Let $f \in H^{-1+s}(\Omega)$, $s \le s_\Omega$, $\boldsymbol{u}_h = (u_h, \boldsymbol{\sigma}_h, \widehat{u}_h, \widehat{\sigma}_h) \in U_h$ DPG approx.

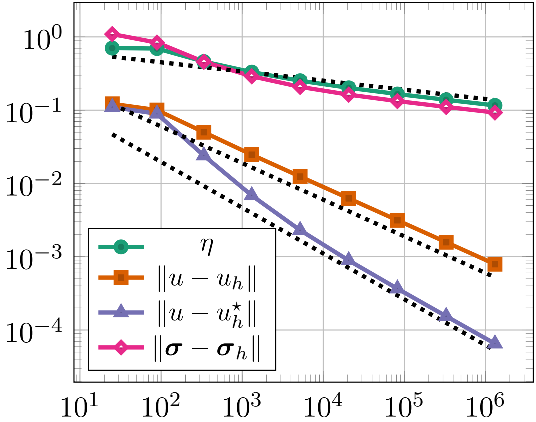
$$\|u - u_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\widehat{u} - \widehat{u}_h\|_{1/2,\mathcal{S}} \lesssim h^s \|f\|_{-1+s}$$

If $Q_h^\star = P_h'$ and Ω convex, then

$$\|u - u_h\| \lesssim h\|f\|_{-1}$$

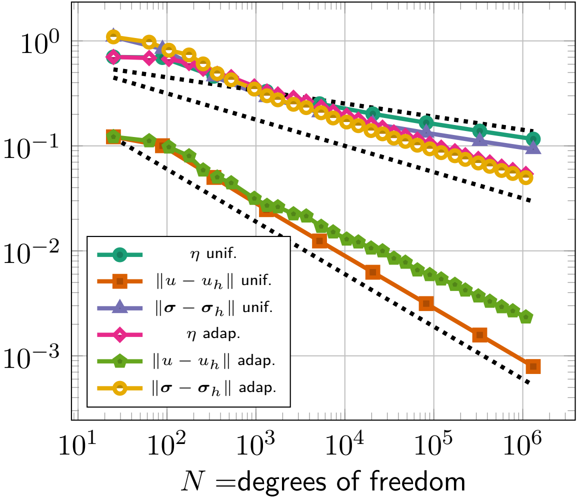
Numerical example

As before for the FOSLS



Dotted lines correspond to $\mathcal{O}(h^{1/4})$, $\mathcal{O}(h)$, and $\mathcal{O}(h^{1+1/4})$

Uniform vs. adaptive



Dotted lines correspond to $\mathcal{O}(N^{-1/8})$, $\mathcal{O}(N^{-1/4})$, and $\mathcal{O}(N^{-1/2})$

Point loads

Dirac distribution at $x_0 \in \Omega$ as force: $(u \notin H^1(\Omega)$ for 2D and 3D$)$

$$-\Delta u = \delta_{x_0}$$

Scott: Finite element convergence for singular data, Numer. Math. '73

Dirac distribution at $x_0 \in \Omega$ as force: ($u \notin H^1(\Omega)$ for 2D and 3D)

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Recall that to compute $P_h' f$ resp. $Q_h f$ we only need to know the actions

$$(f\,, \eta_z), \quad (f\,, \eta_{b,T})$$

These functions are continuous so we replace f by δ_{x_0}

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Lemma (F., Heuer, Karkulik, '22)

$$\|Q_h^\star \delta_{x_0}\| \simeq h^{-n/2}, \qquad \|Q_h^\star \delta_{x_0}\|_{-1} \lesssim \begin{cases} |\log h|^{1/2} & n = 2, \\ h^{-1/2} & n = 3. \end{cases}$$

$$\|(1 - Q_h^\star)\delta_{x_0}\|_{-2} \lesssim h^{2-n/2}$$

Note: The last estimate for $Q_h^\star = Q_h$ is true under mesh condition

Scott: Finite element convergence for singular data, Numer. Math. '73

Regularized MINRES with point loads

DPG method with load $P_h' \delta_{x_0}$

Theorem (F., Heuer, Karkulik, '22)

Suppose Ω *convex,* $\boldsymbol{u}_h = (u_h, \boldsymbol{\sigma}_h, \widehat{u}_h, \widehat{\sigma}_h) \in U_h$ *DPG approx.:*

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FOSLS with load $Q_h\delta_{x_0}$

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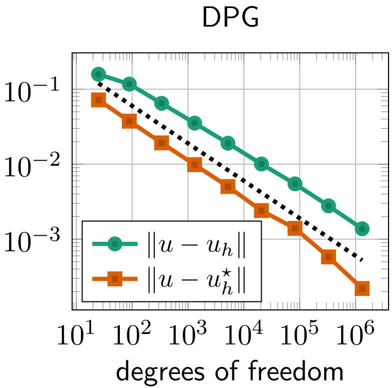
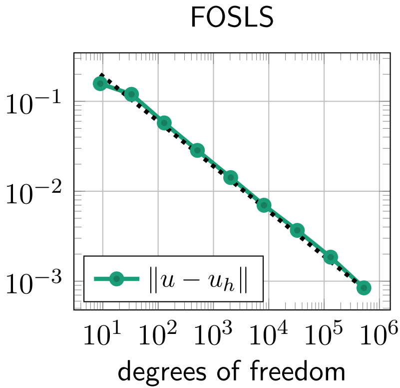
Suppose Ω *convex,* $(u_h, \boldsymbol{\sigma}_h) \in W_h$ *FOSLS approx.:*

$$\|u - u_h\| \lesssim h^{2-n/2}$$

Note: Latter result holds under assumption on mesh:
$s_z = z$ for all interior vertices z, and s_z is the patch center of mass

$\Omega = (-1,1)^2$, $x_0 = (0,0)$.





Conclusion

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Regularized MINRES methods for singular data

- FOSLS and DPG (basic ideas apply to other methods)
- **¡Quasi-optimal!**
- Extends to point loads
- Extends to other PDEs, e.g.,

$$\begin{aligned} -\mathrm{div}\,(\boldsymbol{A}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + cu &= f, \\ u|_{\Gamma_D} &= 0, \\ \boldsymbol{A}\nabla u\cdot\boldsymbol{n}|_{\Gamma_N} &= 0 \end{aligned}$$

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$$

Regularization operators

- $P_h'\colon H^{-1}(\Omega)\to\mathcal{P}^1(\mathcal{T})$ and $Q_h\colon H^{-1}(\Omega)\to\mathcal{P}^0(\mathcal{T})$
- Locally defined and computable
- Bounded in $H^{-s}(\Omega)$, $s\in[0,1]$
- Idempotent on piecewise constants and approximation properties

Thank you for your attention!

- Thomas Führer, Norbert Heuer, Michael Karkulik: **MINRES for second-order PDEs with singular data**, `SINUM`, 60, 2022
- Thomas Führer: **Multilevel decompositions and norms for negative order Sobolev spaces**, `Math. Comp.`, 91, 2021
- Thomas Führer: **Quasi-optimality of FOSLS with H^{-1} loads**, `in preparation`, 2022+

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