Convergence analysis and numerical comparison of adaptive least-squares finite element methods

Dr. Philipp Bringmann

Minimum Residual & Least-Squares Finite Element Methods

Pontificia Universidad Católica de Chile, Santiago

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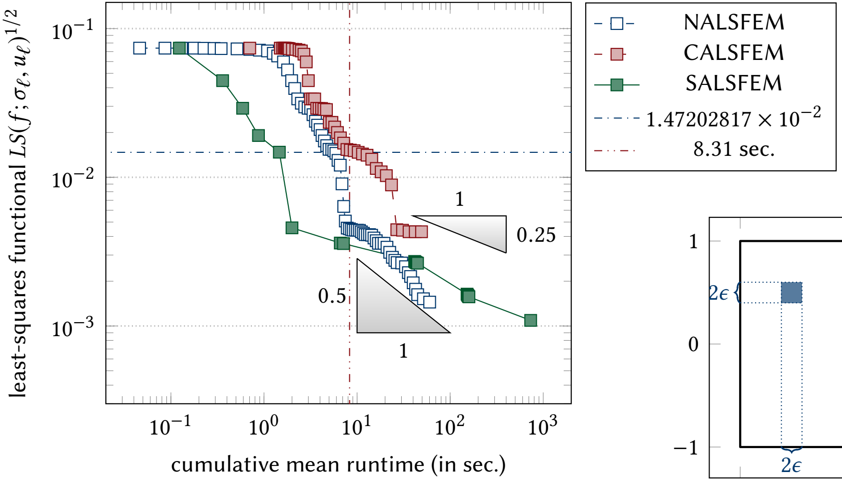
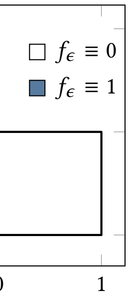
Motivation – Adaptive mesh-refinement





Averaged time measurements on 16 CPUs
(Intel Xeon Platinum 8280L with max. 4.0GHz)

$-\Delta u = f_\epsilon$ in Ω and $u = 0$ on $\partial\Omega$

Outline

Adaptive least-squares FEMs

Least-squares formulation

$\Omega \subset \mathbb{R}^2$ polygonal Lipschitz domain, $\quad f \in L^2(\Omega)$,

$\partial\Omega = \Gamma_\mathrm{D} \cup \Gamma_\mathrm{N}$ into Dirichlet and Neumann boundary Γ_D and Γ_N with $|\Gamma_\mathrm{D}|, |\Gamma_\mathrm{N}| > 0$

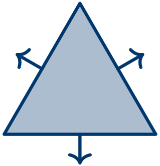
$$\begin{aligned} \text{Minimise} \quad & LS(f;\sigma,u) := \|f + \operatorname{div}\sigma\|^2_{L^2(\Omega)} + \|\sigma - \nabla u\|^2_{L^2(\Omega)} \\ \text{s.t.} \quad & \sigma \in \Sigma := \{\tau \in H(\operatorname{div},\Omega) \ : \ \tau\nu = 0 \text{ on } \Gamma_\mathrm{N}\} \\ & u \in U := \{v \in H^1(\Omega) \ : \ v = 0 \text{ on } \Gamma_\mathrm{D}\} \end{aligned}$$

[Führer-Heuer-Karkulik, *SINUM*, 2022] for right-hand sides in $H^{-1}(\Omega)$

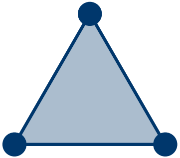
Let $\mathcal{T}$ be regular triangulation of Ω resolving Γ_{D} and Γ_{N} and $k \in \mathbb{N}_0$

$$\begin{aligned} \text{Minimise} \quad & LS(f;\sigma_{\mathrm{LS}},u_{\mathrm{LS}}) := \|f + \operatorname{div}\sigma_{\mathrm{LS}}\|^2_{L^2(\Omega)} + \|\sigma_{\mathrm{LS}} - \nabla u_{\mathrm{LS}}\|^2_{L^2(\Omega)} \\ \text{s.t.} \quad & \sigma_{\mathrm{LS}} \in \Sigma^k(\mathcal{T}) := \{\tau_{\mathrm{RT}} \in RT_k(\mathcal{T}) \ : \ \tau_{\mathrm{RT}}\nu = 0 \text{ on } \Gamma_{\mathrm{N}}\} \\ & u_{\mathrm{LS}} \in U^{k+1}(\mathcal{T}) := \{v_{\mathrm{C}} \in S^{k+1}(\mathcal{T}) \ : \ v_{\mathrm{C}} = 0 \text{ on } \Gamma_{\mathrm{D}}\} \end{aligned}$$

$k = 0$



RT_0



S^1

Built-in a posteriori error estimation

For subsets $\mathcal{M} \subseteq \mathcal{T}$ define the error estimator $\eta_{\mathrm{N}}^2(\mathcal{T}, \mathcal{M}) := \sum_{T \in \mathcal{M}} \eta_{\mathrm{N}}^2(\mathcal{T}, T)$ with

$$\eta_{\mathrm{N}}^2(\mathcal{T}, T) := \|f + \operatorname{div} \sigma_{\mathrm{LS}}\|_{L^2(T)}^2 + \|\sigma_{\mathrm{LS}} - \nabla u_{\mathrm{LS}}\|_{L^2(T)}^2$$

Reliability and efficiency

$$\eta_{\mathrm{N}}^2(\mathcal{T}) := \eta_{\mathrm{N}}^2(\mathcal{T}, \mathcal{T}) = LS(f; \sigma_{\mathrm{LS}}, u_{\mathrm{LS}}) \approx \|\sigma - \sigma_{\mathrm{LS}}\|_{H(\operatorname{div},\Omega)}^2 + \|\nabla(u - u_{\mathrm{LS}})\|_{L^2(\Omega)}^2$$

Asymptotic exactness

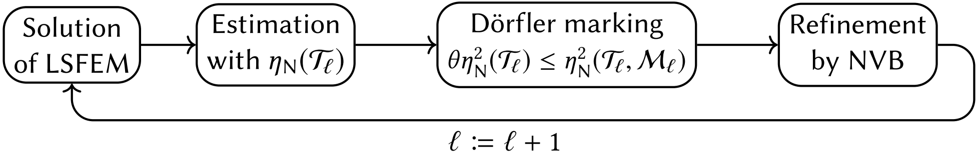
$$LS(f; \sigma_{\mathrm{LS}}, u_{\mathrm{LS}}) / \left(\|\sigma - \sigma_{\mathrm{LS}}\|_{H(\operatorname{div},\Omega)}^2 + \|\nabla(u - u_{\mathrm{LS}})\|_{L^2(\Omega)}^2\right) \to 1 \quad \text{as} \quad h_{\max} \to 0$$

[Carstensen-Storn, *SINUM*, 2018]

Natural adaptive algorithm with collective marking

NALSFEM

Input: initial triangulation $\mathcal{T}_0$ and bulk parameter $0 < \theta \leq 1$



Output: triangulations $\mathcal{T}_\ell$ with solutions σ_ℓ, u_ℓ

[Dörfler, *SINUM*, 1996] [Pfeiler-Praetorius, *Math. Comp.* 2020] [Stevenson, *Math. Comp.* 2008]

Let $\eta_{\mathrm{S}}^2(\mathcal{T}) := \sum_{T\in\mathcal{T}} \eta_{\mathrm{S}}^2(\mathcal{T},T)$, $\mu^2(\mathcal{T}) := \sum_{T\in\mathcal{T}} \mu^2(T)$, and $\operatorname{osc}^2(f,\mathcal{T}) := \sum_{T\in\mathcal{T}} \operatorname{osc}^2(f,T)$ with

$$\begin{aligned}\eta_{\mathrm{S}}^2(\mathcal{T},T) := |T|\Big(\|\operatorname{div}(\sigma_{\mathrm{LS}} - \nabla u_{\mathrm{LS}})\|_{L^2(T)}^2 + \|\operatorname{curl}\sigma_{\mathrm{LS}}\|_{L^2(T)}^2\Big)\\ + |T|^{1/2} \sum_{F\in\mathcal{F}(T)\setminus\mathcal{F}(\Gamma_{\mathrm{D}})} \|[\nabla u_{\mathrm{LS}}\cdot n_F]_F\|_{L^2(F)}^2\\ + |T|^{1/2} \sum_{F\in\mathcal{F}(T)\setminus\mathcal{F}(\Gamma_{\mathrm{N}})} \|[\sigma_{\mathrm{LS}}\cdot t_F]_F\|_{L^2(F)}^2\end{aligned}$$

$$\mu^2(T) := \|(1-\Pi_k)f\|_{L^2(T)}^2 \qquad \operatorname{osc}^2(f,T) := |T|^{1/2}\|(1-\Pi_k)f\|_{L^2(T)}^2$$

[Carstensen-Park, *SINUM*, 2015]

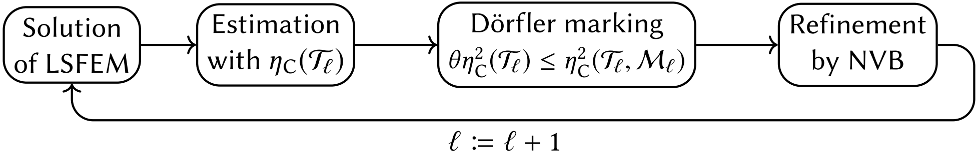
Alternative adaptive algorithm with collective marking

The estimator $\eta_{\mathrm{C}}^2(\mathcal{T},T) := \eta_{\mathrm{S}}^2(\mathcal{T},T) + \mathrm{osc}^2(f,T)$ satisfies

$$\eta_{\mathrm{C}}^2(\mathcal{T}) := \sum_{T\in\mathcal{T}} \eta_{\mathrm{C}}^2(\mathcal{T},T) \approx \|\sigma - \sigma_{\mathrm{LS}}\|_{L^2(\Omega)}^2 + \|\nabla(u - u_{\mathrm{LS}})\|_{L^2(\Omega)}^2 + \mathrm{osc}^2(f,\mathcal{T})$$

CALSFEM

Input: initial triangulation $\mathcal{T}_0$ and bulk parameter $0 < \theta \leq 1$



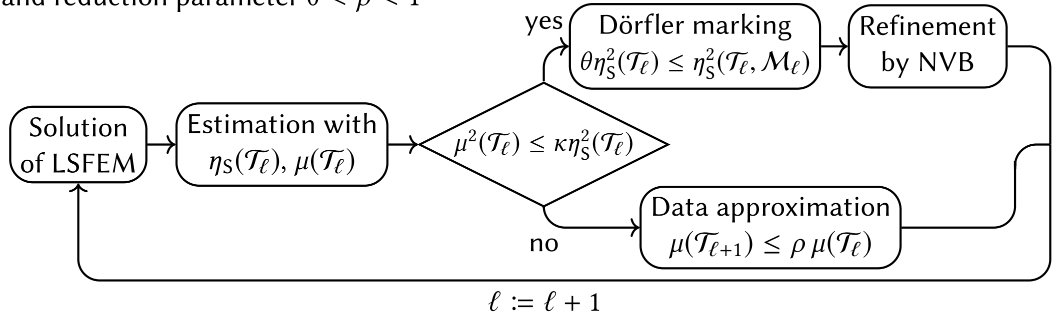
Output: triangulations $\mathcal{T}_\ell$ with solutions σ_ℓ, u_ℓ

[Carstensen, *Math. Comp.* 2020] [Carstensen-Ma, *CAMWA*, 2021]

Alternative adaptive algorithm with separate marking

SALSFEM

Input: initial triangulation $\mathcal{T}_0$, separation parameter $0 < \kappa$, bulk parameter $0 < \theta \leq 1$, and reduction parameter $0 < \rho < 1$



Output: triangulations $\mathcal{T}_\ell$ with solutions σ_ℓ, u_ℓ

[Carstensen-Park, *SINUM*, 2015]

[Binev-DeVore, *Numer. Math.* 2004]
[Carstensen-Rabus, *Math. Comp.* 2011]

Convergence analysis

Plain convergence of NALSFEM

The output $(\sigma_\ell, u_\ell)_\ell$ of NALSFEM satisfies

$$\|\sigma - \sigma_\ell\|^2_{H(\mathrm{div},\Omega)} + \|\nabla(u - u_\ell)\|^2_{L^2(\Omega)} + \eta^2_{\mathrm{N}}(\mathcal{T}_\ell) \to 0 \quad \text{as} \quad \ell \to \infty$$

even for other marking strategies

[Führer-Praetorius, *CAMWA*, 2020; Gantner-Stevenson, *M2AN*, 2021; Siebert, *IMA JNA*, 2011]

For $k = 0$

Assume that the initial triangulation is sufficiently fine in that $f = \Pi_{L+1} f$

There exist a minimal bulk parameter $0 < \theta_0 < 1$ and constants $0 < \rho < 1$ and $0 < \Lambda < \infty$ such that for all $\theta_0 \leq \theta \leq 1$, the modified estimator

$$\tilde{\eta}_{\mathrm{N}}^2(\mathcal{T}_\ell) := LS(f; \sigma_\ell, u_\ell) + \Lambda \, \|(1 - \Pi_\ell)\sigma_\ell\|_{L^2(\Omega)}^2$$

satisfies

$$\tilde{\eta}_{\mathrm{N}}^2(\mathcal{T}_{\ell+1}) \leq \rho \, \tilde{\eta}_{\mathrm{N}}^2(\mathcal{T}_\ell) \quad \text{for all } \ell = L, L+1, \dots$$

[Carstensen-Park-Bringmann, *Numer. Math.* 2017]

Optimal convergence rate of CALSFEM

There exists a maximal bulk parameter $0 < \theta_0 < 1$ such that for all $0 < \theta < \theta_0$ and $0 < s < \infty$, the output $(\sigma_\ell, u_\ell)_\ell$ of CALSFEM satisfies

$$\sup_{N \in \mathbb{N}} (N+1)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta_{\mathrm{C}}(\mathcal{T}) \approx \sup_{\ell \in \mathbb{N}} \big(|\mathcal{T}_\ell| - |\mathcal{T}_0| + 1\big)^s \eta_{\mathrm{C}}(\mathcal{T}_\ell)$$

[Carstensen, *Math. Comp.* 2020] [Carstensen-Ma, *CAMWA*, 2021]

Proof by axioms of adaptivity [Carstensen et al. *CAMWA*, 2014]

Optimal convergence rate of SALSFEM

There exist a maximal bulk parameter $0 < \theta_0 < 1$ and a maximal separation parameter $0 < \kappa_0 \leq \infty$ such that for all $0 < \theta < \theta_0$, $0 < \kappa < \kappa_0$, $0 < \rho < 1$, and $0 < s < \infty$, the output $(\sigma_\ell, u_\ell)_\ell$ of SALSFEM satisfies

$$\sup_{N \in \mathbb{N}} (N+1)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \left(\eta_{\mathrm{S}}^2(\mathcal{T}) + \mu^2(\mathcal{T})\right)^{1/2} \approx \sup_{\ell \in \mathbb{N}} \left(|\mathcal{T}_\ell| - |\mathcal{T}_0| + 1\right)^s \left(\eta_{\mathrm{S}}^2(\mathcal{T}_\ell) + \mu^2(\mathcal{T}_\ell)\right)^{1/2}$$

Proof by axioms of adaptivity [Carstensen-Rabus, *SINUM*, 2017]

Unified proof for Poisson problem, Stokes equations, and linear elasticity in 2D and 3D with inhomogeneous boundary conditions and arbitrary polynomial degree $k \in \mathbb{N}_0$ presented in

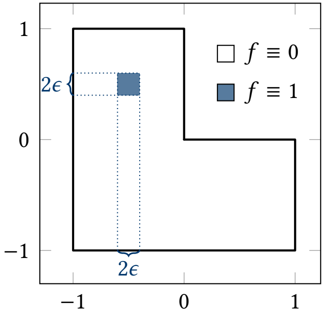
[Bringmann, *PhD thesis*, 2021] [Bringmann, *J. Numer. Math.* 2022]

Numerical comparison

2D benchmark example with microstructure

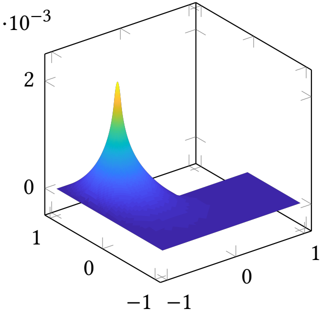
Poisson model problem with homogeneous Dirichlet boundary conditions

$$f_\epsilon \in L^2(\Omega) \quad \text{with} \quad \epsilon > 0 \quad \text{and} \quad f_\epsilon(x) := \begin{cases} 1 & \text{if } |x_1 + \frac{1}{2}| \leq \epsilon \text{ and } |x_2 - \frac{1}{2}| \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$



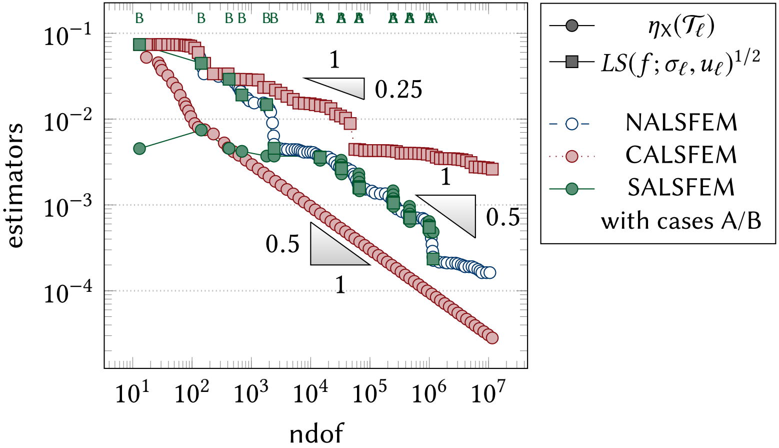
L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1)^2$

[Rabus, *J. Numer. Math.* 2015, Sect. 3.4]



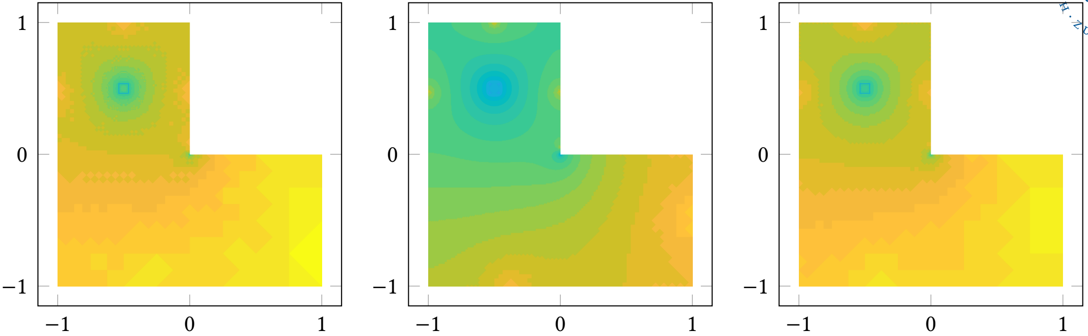
Approximation to solution of $-\Delta u = f_\epsilon$

[Sutherland-Hodgman, *Commun. ACM*, 1974]



$$\epsilon = 3^{-3}, \quad \theta = 0.3, \quad \kappa = 1, \quad \rho = 0.8$$

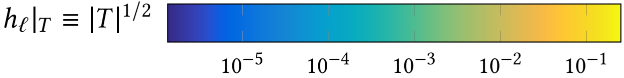
Comparison of meshes



NALSFEM (575 130 triangles)

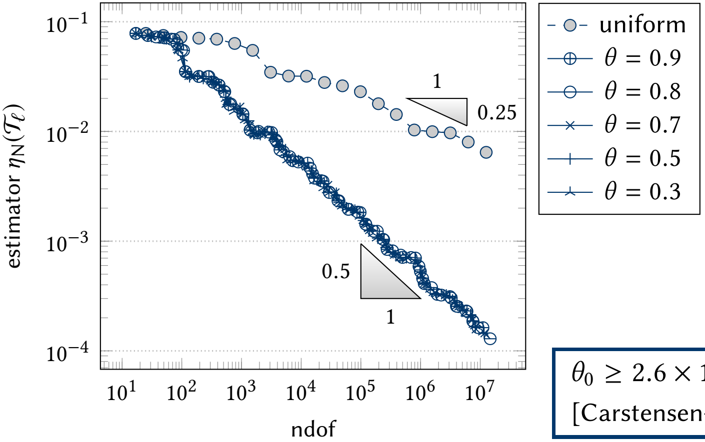
CALSFEM (551 344 triangles)

SALSFEM (580 208 triangles)



$$\epsilon = 3^{-3}, \quad \theta = 0.3, \quad \kappa = 1, \quad \rho = 0.8$$

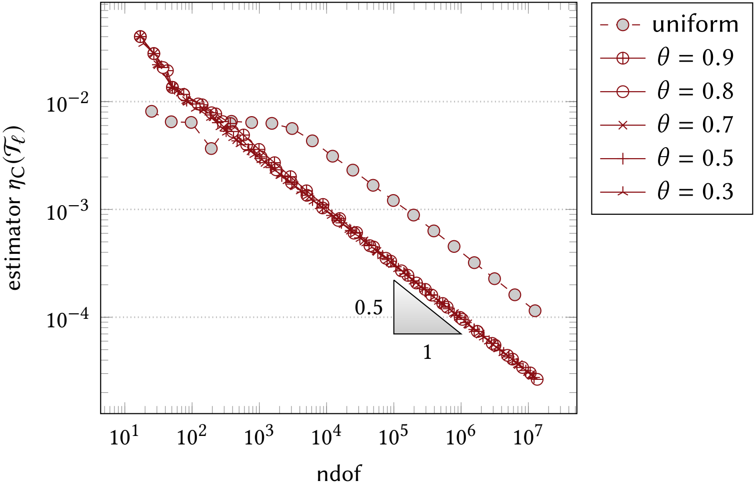
Choice of bulk parameter θ



NALSFEM $\quad \epsilon = 3^{-3}$

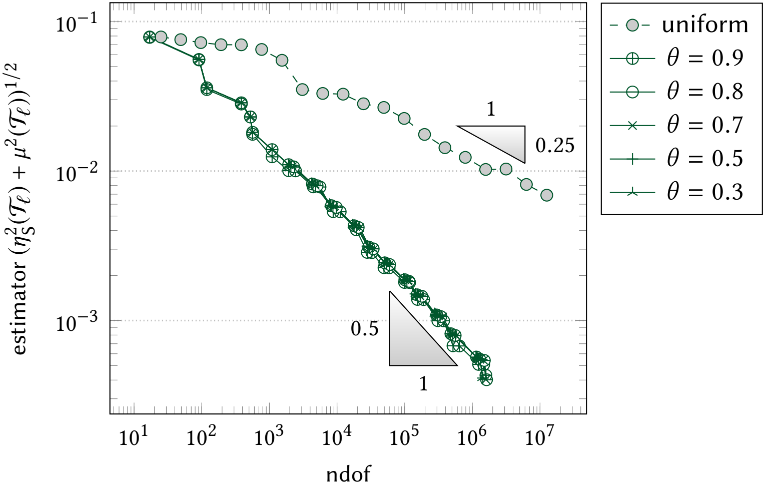
$\theta_0 \geq 2.6 \times 10^{-6}$ for Courant FEM
[Carstensen-Hellwig, *CMAM*, 2018]

Choice of bulk parameter θ

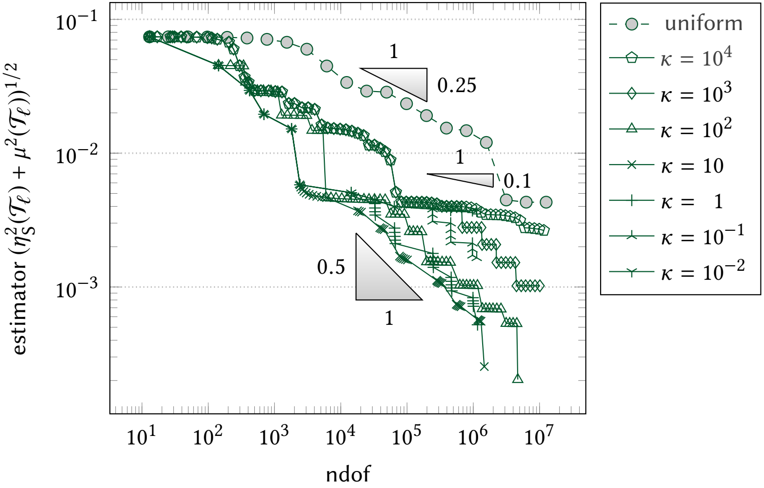


CALSFEM $\epsilon = 3^{-3}$

Choice of bulk parameter θ

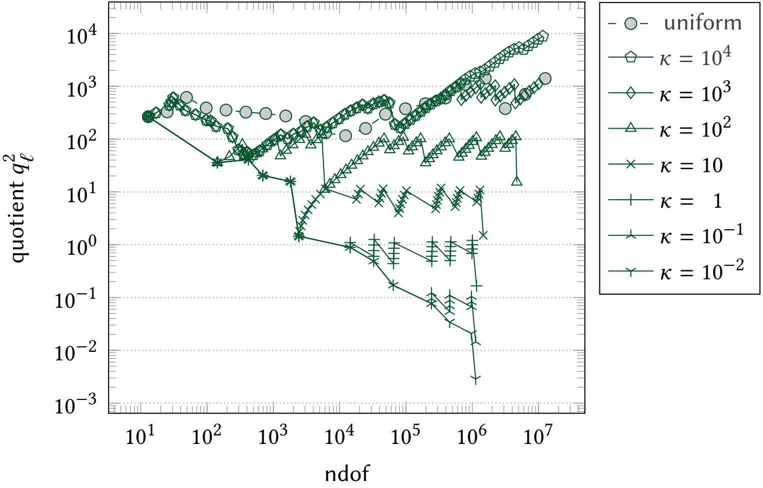


SALSFEM $\quad \epsilon = 3^{-3}, \quad \kappa = 10, \quad \rho = 0.8$



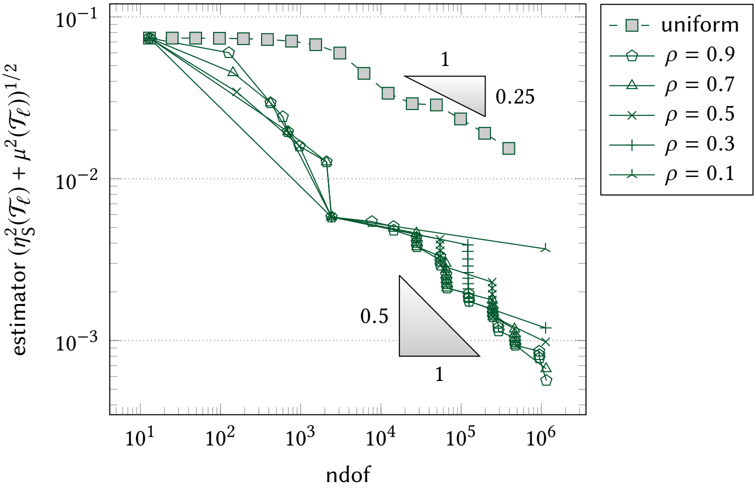
$$\epsilon = 3^{-3}, \quad \theta = 0.3, \quad \rho = 0.8$$

Choice of separation parameter κ in SALSFEM



$q_\ell^2 := \mu^2(\mathcal{T}_\ell)/\eta_S^2(\mathcal{T}_\ell)$ $\qquad \epsilon = 3^{-3}, \quad \theta = 0.3, \quad \rho = 0.8$

Choice of reduction parameter ρ



$$\epsilon = 3^{-3}, \quad \theta = 0.3, \quad \kappa = 1$$

Conclusion

Choice of method

SALSFEM for guaranteed optimal convergence rates

NALSFEM for efficient alternative

CALSFEM for efficient method with optimal convergence rates in weaker norm

Choice of parameters

bulk parameter $0.3 \leq \theta \leq 0.7$

separation parameter $\kappa := q_\ell^2/10$ for $\ell = 0$ or small $\ell > 0$ (e.g., from uniform refinement)

reduction parameter $0.3 \leq \rho \leq 0.8$

Thank you for your attention!

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DFG

SPP 1748

Publications

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P. Bringmann. *Computational competition of three adaptive least-squares finite element schemes.* Submitted. Preprint available at arXiv:2209.06028. 2022.

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Publications

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Poisson model problem in 2D, homogeneous Dirichlet BC, lowest-order discretisation

[Bringmann-Carstensen, *Numer. Math.* 2017]
Stokes equations in 2D, inhomogeneous Dirichlet BC, lowest-order discretisation

[Bringmann-Carstensen, *CAMWA*, 2017]
Stokes equations in 2D, inhomogeneous Dirichlet BC, higher-order discretisation

[Bringmann-Carstensen-Starke, *SINUM*, 2018]
Linear elasticity in 3D, mixed inhomogeneous Neumann BC, lowest-order discretisation

[Bringmann, *PhD thesis*, 2021]
Generalised model problem in 3D, mixed inhomogeneous BC, higher-order discretisation

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