On the computation of Maxwell’s eigenvalues with nodal elements

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Outline

The Maxwell eigenvalue problem

Ampère and Faraday's laws: find resonance frequencies $\omega \in \mathbb{R}$ (with $\omega \neq 0$) and electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} &= i\omega\mu\mathbf{H} && \text{in } \Omega \\ \mathbf{curl}\,\mathbf{H} &= -i\omega\varepsilon\mathbf{E} && \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} &= 0 && \text{on } \partial\Omega \\ \mathbf{H} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

$\omega \neq 0$ gives divergence conditions

$$\begin{aligned} \operatorname{div} \varepsilon\mathbf{E} &= 0 && \text{in } \Omega \\ \operatorname{div} \mu\mathbf{H} &= 0 && \text{on } \Omega \end{aligned}$$

It is then standard to eliminate one field and to obtain the **curl curl** problem

The **curl curl** problem

Eliminate $\mathbf{H}$ and take $\mathbf{u} = \mathbf{E}$, $\lambda = \omega^2$

$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = \lambda\varepsilon\mathbf{u} & \text{in } \Omega \\ \operatorname{div}(\varepsilon\mathbf{u}) = 0 & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Well-known and intensively studied problem. Special (*edge*) finite elements required for its approximation.

Edge elements are discrete 1-forms

For ease of presentation, we take $\mu = \varepsilon = 1$ and simple topology from now on.

Standard formulation

The standard variational formulation reads

$$
\begin{aligned}
&\lambda \in \mathbb{R},\ \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : \\
&(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \qquad && \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\
&(\mathbf{u}, \mathbf{grad}\,\phi) = 0 && \forall \phi \in H_0^1
\end{aligned}
$$

The most commonly used variational formulation is based on the replacement of the divergence free constraint by the condition $\lambda \neq 0$

$$
\begin{aligned}
&\lambda \in \mathbb{R} \setminus 0,\ \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : \\
&(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl})
\end{aligned}
$$

The kernel $\lambda = 0$ corresponds to the infinite dimensional space $\mathbf{grad}\, H_0^1$ (N.B.: Helmholtz decomposition)

Maxwell's eigenvalues

Find $\lambda \neq 0$ such that

$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$:
$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl})$

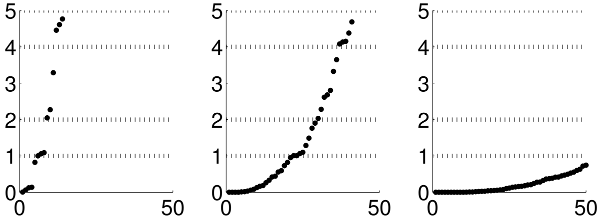
We take $\Omega = (0, \pi) \times (0, \pi)$ so that the exact solutions are

$\lambda_{mn} = m^2 + n^2 = 1, 1, 2, 4, 4, 5, 5, \dots$
$\mathbf{u}_{mn}(x, y) = \mathbf{curl}(\cos(mx)\cos(ny))$

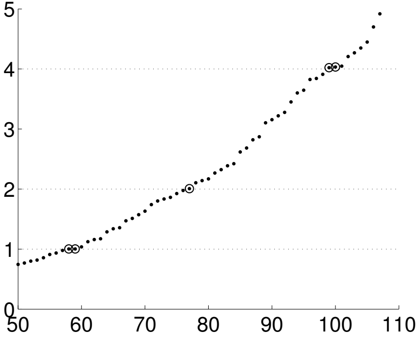
N.B.

The approximation has to take into account the infinite dimensional kernel of the continuous problems corresponding to the gradients of $H_0^1(\Omega)$

Unstructured mesh ($N = 4, 8, 16$)

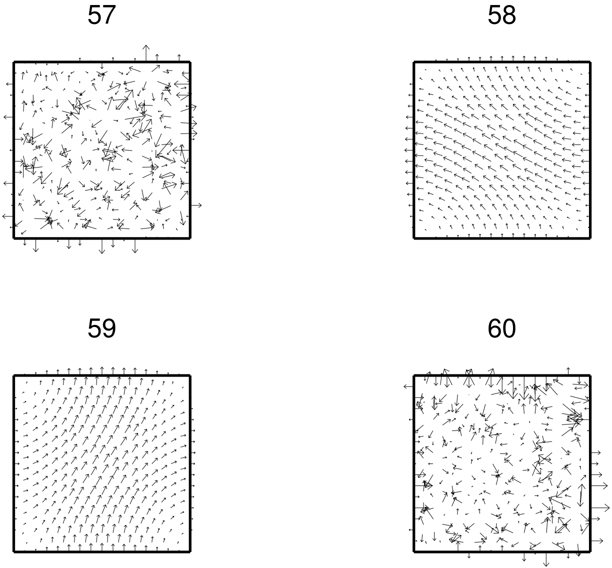


Zoom for $N = 16$



Standard P1 elements (cont'ed)

Some eigenfunctions



Standard P1 elements (cont'd)

Some more eigenfunctions

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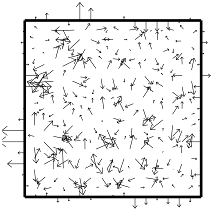

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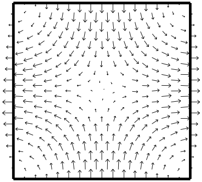

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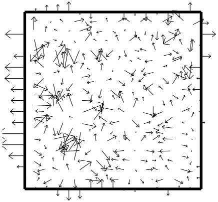


Commuting diagram property (de Rham complex)

⟨Douglas–Roberts ’82⟩ ⟨Bossavit ’88⟩ ⟨Arnold ’02⟩

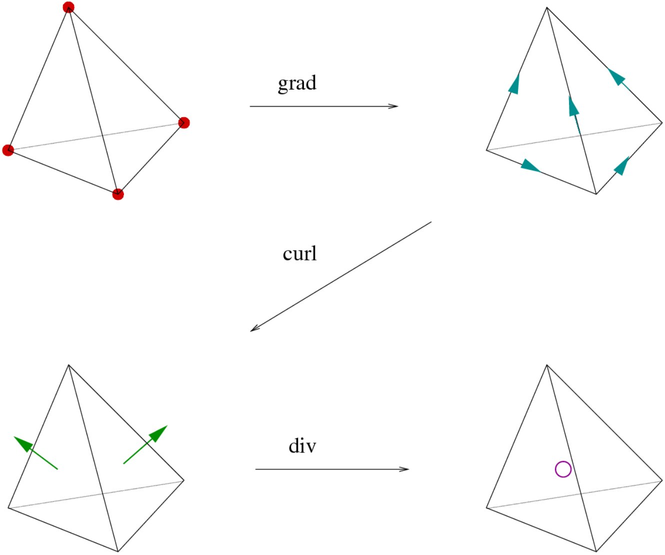
$Q \subset H^1_0,\ V \subset \mathbf{H}_0(\mathbf{curl}),\ U \subset \mathbf{H}_0(\mathrm{div}),\ S \subset L^2/\mathbb{R}$

$$
\begin{array}{ccccccccc}
0 \to & Q & \xrightarrow{\mathbf{grad}} & V & \xrightarrow{\mathbf{curl}} & U & \xrightarrow{\mathrm{div}} & S & \to 0 \\
& \downarrow \Pi_k^Q & & \downarrow \Pi_k^V & & \downarrow \Pi_k^U & & \downarrow \Pi_k^S & \\
0 \to & Q_k & \xrightarrow{\mathbf{grad}} & V_k & \xrightarrow{\mathbf{curl}} & U_k & \xrightarrow{\mathrm{div}} & S_k & \to 0
\end{array}
$$

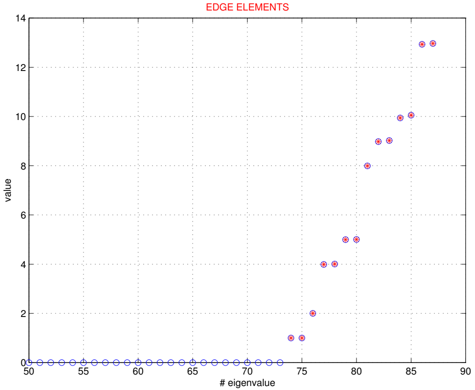
In 2D the complex reduces to

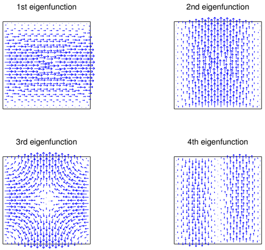
$$
\begin{array}{ccccccc}
0 \to & Q & \xrightarrow{\mathbf{grad}} & V & \xrightarrow{\mathbf{curl}} & S & \to 0 \\
& \downarrow \Pi_k^Q & & \downarrow \Pi_k^V & & \downarrow \Pi_k^S & \\
0 \to & Q_k & \xrightarrow{\mathbf{grad}} & V_k & \xrightarrow{\mathbf{curl}} & S_k & \to 0
\end{array}
$$

Edge elements



Nice convergence on general meshes





Optimal rate of convergence

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	0.9702	0.9923 (2.0)	0.9981 (2.0)	0.9995 (2.0)	0.9999 (2.0)
1	0.9960	0.9991 (2.2)	0.9998 (2.1)	0.9999 (2.0)	1.0000 (2.0)
2	2.0288	2.0082 (1.8)	2.0021 (2.0)	2.0005 (2.0)	2.0001 (2.0)
4	3.7227	3.9316 (2.0)	3.9829 (2.0)	3.9957 (2.0)	3.9989 (2.0)
4	3.7339	3.9325 (2.0)	3.9829 (2.0)	3.9957 (2.0)	3.9989 (2.0)
5	4.7339	4.9312 (2.0)	4.9826 (2.0)	4.9956 (2.0)	4.9989 (2.0)
5	5.1702	5.0576 (1.6)	5.0151 (1.9)	5.0038 (2.0)	5.0010 (2.0)
8	7.4306	8.1016 (2.5)	8.0322 (1.7)	8.0084 (1.9)	8.0021 (2.0)
9	7.5231	8.6292 (2.0)	8.9061 (2.0)	8.9764 (2.0)	8.9941 (2.0)
9	7.9586	8.6824 (1.7)	8.9211 (2.0)	8.9803 (2.0)	8.9951 (2.0)
zero	9	49	225	961	3969
dof	40	176	736	3008	12160

A priori analysis

A posteriori analysis

Convergence of AFEM ⟨B.–Gastaldi ’20⟩

Outline

Some introductory examples

Maxwell's eigenvalues and finite element least squares

Joint work with F. Bertrand and L. Gastaldi

Numerical analysis in 2D

Numerical tests in 2D

The 3D case

First order least squares approach (doesn't work)

Let's start with the source problem

Description in 3D, some (simplifying) changes are needed in 2D

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \mathbf{f} & \text{in } \Omega \quad (\operatorname{div} \mathbf{f} = 0) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

First order system

$$\begin{cases} \mathbf{p} = \mathbf{curl}\,\mathbf{u} \\ \mathbf{curl}\,\mathbf{p} = \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \end{cases}$$

Minimization of

$$\mathcal{F}(\mathbf{q}, \mathbf{v}) = \|\mathbf{q} - \mathbf{curl}\,\mathbf{v}\|_0^2 + \|\,\mathbf{curl}\,\mathbf{v} - \mathbf{f}\|_0^2 + \|\operatorname{div} \mathbf{v}\|_0^2$$

Singular solutions cannot be approximated

$\mathbf{H}^1 \cap \boldsymbol{H}_0(\mathbf{curl})$ is a closed subspace of $\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div})$

⟨Costabel '91⟩

Possible cure: weighted least-squares formulation

⟨Fix–Stephan '82, Cox–Fix '84⟩

Another possible cure: use a discrete divergence operator and edge elements ⟨Bochev–Peterson–Siefert '11⟩

Alternative first order formulation

⟨B.–Fernandes–Gastaldi–Perugia '99⟩

Take any potential $\mathbf{g}$ such that $\mathbf{curl}\,\mathbf{g} = \mathbf{f}$ and solve

$$\begin{cases} \mathbf{u} = \mathbf{curl}\,\mathbf{p} & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{u} = \mathbf{g} & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

NB: the uniqueness of $\mathbf{p}$ is guaranteed if it is sought in

$$\boldsymbol{H}_0(\mathrm{div}^0) = \mathbf{curl}(\boldsymbol{H}_0(\mathbf{curl}))$$

that is,

$$\begin{cases} \mathrm{div}\,\mathbf{p} = 0 & \text{in } \Omega \\ \mathbf{p} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

NB2: in 2D the condition has the simpler form $\int_\Omega p = 0$

LS minimization principle (alternative formulation)

3D

Minimization of

$$\mathcal{F}(\mathbf{v},\mathbf{q}) = \|\mathbf{v} - \mathbf{curl}\,\mathbf{q}\|_0^2 + \|\,\mathbf{curl}\,\mathbf{v} - \mathbf{g}\|_0^2$$

in

$$\boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}^0)$$

2D

Minimization of

$$\mathcal{F}(\mathbf{v},q) = \|\mathbf{v} - \mathbf{curl}\,q\|_0^2 + \|\,\mathrm{rot}\,\mathbf{v} - g\|_0^2$$

in

$$\boldsymbol{H}_0(\mathrm{rot}) \times H(\mathbf{curl}) \cap L_0^2(\Omega)$$

Variational formulation (3D)

Find $\mathbf{u} \in \boldsymbol{H}_0(\mathbf{curl})$ and $\mathbf{p} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}^0) =: \mathbf{Q}$ such that

$$\begin{cases} (\mathbf{u}, \mathbf{v}) + (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) - (\mathbf{curl}\,\mathbf{p}, \mathbf{v}) = (\mathbf{g}, \mathbf{curl}\,\mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ -(\mathbf{u}, \mathbf{curl}\,\mathbf{q}) + (\mathbf{curl}\,\mathbf{p}, \mathbf{curl}\,\mathbf{q}) = 0 & \forall \mathbf{q} \in \mathbf{Q} \end{cases}$$

Corresponding eigenvalue problem formulation: find λ and $\mathbf{p} \in \mathbf{Q}$ with $\mathbf{p} \neq 0$ such that for some $\mathbf{u} \in \boldsymbol{H}_0(\mathbf{curl})$

$$\begin{cases} (\mathbf{u}, \mathbf{v}) + (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) - (\mathbf{curl}\,\mathbf{p}, \mathbf{v}) = \lambda(\mathbf{p}, \mathbf{curl}\,\mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ -(\mathbf{u}, \mathbf{curl}\,\mathbf{q}) + (\mathbf{curl}\,\mathbf{p}, \mathbf{curl}\,\mathbf{q}) = 0 & \forall \mathbf{q} \in \mathbf{Q} \end{cases}$$

Variational formulation (2D)

Find $\mathbf{u} \in \boldsymbol{H}_0(\mathrm{rot})$ and $p \in H(\mathbf{curl}) \cap L^2_0(\Omega)$ such that

$$\begin{cases} (\mathbf{u}, \mathbf{v}) + (\mathrm{rot}\,\mathbf{u}, \mathrm{rot}\,\mathbf{v}) - (\mathbf{curl}\, p, \mathbf{v}) = (g, \mathrm{rot}\,\mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{H}_0(\mathrm{rot}) \\ -(\mathbf{u}, \mathbf{curl}\, q) + (\mathbf{curl}\, p, \mathbf{curl}\, q) = 0 & \forall q \in H(\mathbf{curl}) \cap L^2_0(\Omega) \end{cases}$$

Corresponding eigenvalue problem formulation: find λ and $p \in H(\mathbf{curl}) \cap L^2_0(\Omega)$ with $p \neq 0$ such that for some $\mathbf{u} \in \boldsymbol{H}_0(\mathrm{rot})$

$$\begin{cases} (\mathbf{u}, \mathbf{v}) + (\mathrm{rot}\,\mathbf{u}, \mathrm{rot}\,\mathbf{v}) - (\mathbf{curl}\, p, \mathbf{v}) = \lambda(p, \mathrm{rot}\,\mathbf{v}) & \forall \mathbf{v} \in \boldsymbol{H}_0(\mathrm{rot}) \\ -(\mathbf{u}, \mathbf{curl}\, q) + (\mathbf{curl}\, p, \mathbf{curl}\, q) = 0 & \forall q \in H(\mathbf{curl}) \cap L^2_0(\Omega) \end{cases}$$

Neumann Laplace

This is a LS Neumann eigenvalue problem for the Laplace operator with the usual isomorphisms between **grad**, **curl** and div, rot

Notice that $H(\mathbf{curl}) = H^1$

Analysis of the continuous problem

By mimicking the analysis of the FOSLS formulation for the Laplace problem, we can see that the underlying bilinear form is coercive both in 2D and in 3D

$$\|\mathbf{v} - \mathbf{curl}\,\mathbf{q}\|_0^2 + \|\,\mathbf{curl}\,\mathbf{v}\|_0^2 \geq \alpha\|\mathbf{v}\|_{\mathbf{curl}}^2 + \|\mathbf{q}\|_{\mathbf{curl}}^2$$

$$\|\mathbf{v} - \mathbf{curl}\,q\|_0^2 + \|\,\mathrm{rot}\,\mathbf{v}\|_0^2 \geq \alpha\|\mathbf{v}\|_{\mathrm{rot}}^2 + \|q\|_1^2$$

Poincaré–Friedrichs inequalities

Crucial estimates for the proof are

$$\|\mathbf{q}\|_0 \leq C\|\,\mathbf{curl}\,\mathbf{q}\|_0 \quad \forall\mathbf{q} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}^0)$$

$$\|q\|_0 \leq C\|\,\mathbf{curl}\,q\|_0 \quad \forall q \in H(\mathbf{curl}) \cap L_0^2(\Omega)$$

Outline

Some introductory examples

Maxwell's eigenvalues and finite element least squares

Joint work with F. Bertrand and L. Gastaldi

Numerical analysis in 2D

Numerical tests in 2D

The 3D case

Conforming approximation

Let's start with the 2D case

$$\|\mathbf{v} - \mathbf{curl}\, q\|_0^2 + \|\operatorname{rot}\mathbf{v}\|_0^2 \geq \alpha\|\mathbf{v}\|_{\mathrm{rot}}^2 + \|q\|_1^2 \qquad \begin{array}{l}\forall \mathbf{v} \in \boldsymbol{H}_0(\mathrm{rot}) \\ \forall q \in H(\mathbf{curl}) \cap L_0^2(\Omega)\end{array}$$

Take $V_h \subset \boldsymbol{H}_0(\mathrm{rot})$ and $Q_h \subset H(\mathbf{curl}) \cap L_0^2(\Omega)$ and find λ_h and $p_h \in Q_h$ with $p_h \neq 0$ such that for some $\mathbf{u}_h \in V_h$

$$\begin{cases} (\mathbf{u}_h, \mathbf{v}) + (\operatorname{rot}\mathbf{u}_h, \operatorname{rot}\mathbf{v}) - (\mathbf{curl}\, p_h, \mathbf{v}) = \lambda_h(p_h, \operatorname{rot}\mathbf{v}) & \forall \mathbf{v} \in V_h \\ -(\mathbf{u}_h, \mathbf{curl}\, q) + (\mathbf{curl}\, p_h, \mathbf{curl}\, q) = 0 & \forall q \in Q_h \end{cases}$$

Zero mean value of p

The condition $Q_h \subset L_0^2(\Omega)$ can be easily imposed, for instance, by the use of a one dimensional Lagrange multiplier

Numerical aspects (useful in view of 3D extension)

$$\begin{pmatrix} \mathsf{A} & -\mathsf{B}^\top \\ -\mathsf{B} & \mathsf{C} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = \lambda_h \begin{pmatrix} 0 & \mathsf{B}^\top \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}$$

Schur complement $\mathsf{Cy} = (\lambda + 1)\mathsf{BA}^{-1}\mathsf{B}^\top \mathsf{y}$

Spectrum characterization

- $\lambda_h \in \mathbb{R}$ corresponding to $\mathrm{range}(B)$
- $\lambda_h = \infty$ corresponding to $\dim V_h + \dim(\ker B^\top)$

Ignoring the constraint $Q_h \subset L^2_0(\Omega)$

There is a one dimensional subspace of Q_h for which $\mathsf{Cy} = 0$ and $\mathsf{B}^\top \mathsf{y} = 0$ giving a degenerate eigenvalue with $\mathsf{x} = 0$

NB: the degenerate eigenvalue can be easily spotted: the corresponding p is constant and $\mathbf{u}$ vanishes

Convergence analysis

Standard analysis (coercivity and continuity) gives the quasi-optimal error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{rot}}+\|p-p_h\|_1\le\inf_{\substack{\mathbf{v}\in V_h\\ q\in Q_h}}\left(\|\mathbf{u}-\mathbf{v}\|_{\mathrm{rot}}+\|p-q\|_1\right)$$

Convergence in norm

Necessary and sufficient condition for the convergence of eigensolutions (and absence of spurious solutions) is the convergence in norm of the solution operator corresponding to the variable p

Does energy estimate imply convergence in norm?

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{rot}} + \|p - p_h\|_1 \leq C \inf_{\substack{\mathbf{v}\in V_h \\ q\in Q_h}} \left(\|\mathbf{u} - \mathbf{v}\|_{\mathrm{rot}} + \|p - q\|_1\right)$$

We want

$$\|Tg - T_h g\|_{\boxed{?}} \leq \rho(h)\|g\|_{\boxed{?}} \quad \forall g \in \boxed{?}$$

with $Tg = p$ and $T_h g = p_h$

We explored to possible choices:

- $\boxed{?} = H^1(\Omega)$
- $\boxed{?} = L^2(\Omega)$

Convergence in the $H^1(\Omega)$ norm (edge elements)

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{rot}}+\|p-p_h\|_1\le C\inf_{\substack{\mathbf{v}\in V_h\\ q\in Q_h}}\left(\|\mathbf{u}-\mathbf{v}\|_{\mathrm{rot}}+\|p-q\|_1\right)$$

- $\inf\limits_{q\in Q_h}\|p-q\|_1\le Ch^s\|p\|_{1+s}\le Ch^s\|g\|_1$
  for any reasonable choice of Q_h and $s>0$ as long as $p\in H^{1+s}(\Omega)$
- $\inf\limits_{\mathbf{v}\in V_h}\|\mathbf{u}-\mathbf{v}\|_{\mathrm{rot}}\le Ch^s(\|\mathbf{u}\|_s+\|\operatorname{rot}\mathbf{u}\|_s)\le Ch^s\|g\|_1$
  if V_h is a space based on Nédélec elements and $0<s\le 1$
  (note that $\operatorname{rot}\mathbf{u}=g$ and $\mathbf{curl}\,g=f$)

General results

This analysis works under very general hypotheses on the domain (also with varying coefficients)

Convergence in the $H^1(\Omega)$ norm (nodal elements)

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{rot}} + \|p-p_h\|_1 \le C \inf_{\substack{\mathbf{v}\in V_h\\ q\in Q_h}} \left(\|\mathbf{u}-\mathbf{v}\|_{\mathrm{rot}} + \|p-q\|_1\right)$$

- $\inf_{q\in Q_h} \|p-q\|_1 \le Ch^s\|p\|_{1+s} \le Ch^s\|g\|_1$
  for any reasonable choice of Q_h and $s>0$ as long as $p\in H^{1+s}(\Omega)$
- $\inf_{\mathbf{v}\in V_h} \|\mathbf{u}-\mathbf{v}\|_{\mathrm{rot}} \le \inf_{\mathbf{v}\in V_h} \|\mathbf{u}-\mathbf{v}\|_1 \le Ch^s\|\mathbf{u}\|_{1+s} \le Ch^s\|g\|_1$
  if V_h is based on nodal elements, we need a suitable regularity shift theorem for the solution of the Neumann Laplace with $g\in H^1(\Omega)$

And if the problem is singular?

Energy estimate doesn't imply the convergence in the $H^1(\Omega)$ norm if the solution is nonsmooth

Convergence in the $L^2(\Omega)$ norm?

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{rot}} + \|p - p_h\|_1 \le C \inf_{\substack{\mathbf{v}\in V_h \\ q\in Q_h}} \left(\|\mathbf{u} - \mathbf{v}\|_{\mathrm{rot}} + \|p - q\|_1\right)$$

Lack of compactness

$$\boxed{\mathrm{rot}\,\mathbf{u} = g \text{ and } g \in L^2(\Omega)}$$

doesn't allow any rate of convergence out of

$$\boxed{\inf_{\mathbf{v}\in V_h} \|\mathbf{u} - \mathbf{v}\|_{\mathrm{rot}}}$$

Can a duality argument help?

⟨Bertrand–B. ’21⟩
⟨Bertrand–B. ’21⟩
⟨Alzaben–Bertrand–B. ’22⟩

We get, for all $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$,

$$\|p - p_h\|_0^2 \le C(\|\boldsymbol{\chi} - \mathbf{v}_h\|_{\mathrm{rot}} + \|\varphi - q_h\|_1)(\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{rot}} + \|p - p_h\|_1)$$

where $(\boldsymbol{\chi}, \varphi)$ is the solution of the dual problem with right hand side $p - p_h$

Need to estimate $\|\boldsymbol{\chi} - \mathbf{v}_h\|_{\mathrm{rot}}$ knowing that $\boldsymbol{\chi} \in \mathbf{H}^s(\Omega)$ and $\operatorname{div} \boldsymbol{\chi} \in H^{1+s}(\Omega)$ with $0 < s \le 1$

It doesn’t help for nodal elements

The duality argument works for edge elements only

Other refined L^2-estimates don’t help

⟨Manteuffel–McCormick–Pflaum ’03⟩
⟨Cai–Ku ’06–’10⟩
⟨Ku ’08–’11⟩

Summary of the theory in 2D

- Nédélec elements: optimal convergence in general
- Nodal elements: optimal convergence if the solution of the Neumann Laplace problem is in $H^{2+s}(\Omega)$ ($s > 0$) when $g \in H^1(\Omega)$
- No theory for nodal elements in case of singular problems

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2D square: edge elements

Uniform mesh

16 × 16		32 × 32		64 × 64	
Exact	Computed	Exact	Computed	Exact	Computed
1.	1.00512640	1.	1.00128343	1.	1.00032102
1.	1.00688870	1.	1.00171910	1.	1.00042963
2.	2.02911921	2.	2.00728279	2.	2.00182127
4.	4.08931462	4.	4.02212483	4.	4.00552032
4.	4.08935025	4.	4.02213566	4.	4.00552149
5.	5.12234583	5.	5.03065743	5.	5.00767193
5.	5.20636265	5.	5.05091345	5.	5.01268924
8.	8.44571717	8.	8.11235762	8.	8.02814799
9.	9.44432180	9.	9.10868778	9.	9.02703064
9.	9.46221564	9.	9.11272110	9.	9.02801496

2D square: nodal elements

Nonuniform mesh

16×16		32×32		64×64	
Exact	Computed	Exact	Computed	Exact	Computed
1.	1.00308661	1.	1.00076643	1.	1.00019230
1.	1.00314974	1.	1.00077972	1.	1.00019360
2.	2.01213949	2.	2.00304153	2.	2.00075083
4.	4.04778986	4.	4.01184746	4.	4.00294965
4.	4.04883705	4.	4.01196192	4.	4.00296933
5.	5.07515783	5.	5.01852404	5.	5.00458151
5.	5.07794450	5.	5.01887591	5.	5.00464385
8.	8.19242424	8.	8.04742821	8.	8.01175211
9.	9.24253524	9.	9.05921486	9.	9.01476227
9.	9.24949307	9.	9.06046052	9.	9.01489118

Uniform mesh

Exact	16×16	32×32	64×64
1.47562182408	1.53919587580	1.50024873303	1.48531505072
3.53403136678	3.56428398038	3.54178408609	3.53600223281
9.86960440109	10.0746335908	9.92009554943	9.88218010845
9.86960440109	10.0766112653	9.92059624585	9.88230585305
11.3894793979	11.7010251236	11.4665124870	11.4087142321

Nonuniform mesh

Exact	16×16	32×32	64×64
1.47562182408	1.53125517054	1.49930065219	1.49347538050
3.53403136678	3.55205621308	3.53840487118	3.53530530720
9.86960440109	9.99854787751	9.90031682924	9.87725294330
9.86960440109	9.99854787751	9.90151500253	9.87752148776
11.3894793979	11.5606140486	11.4295987125	11.3997480754

2D cracked square (slit domain): nodal elements

Nonuniform mesh

Exact	16×16	32×32	64×64
1.03407400850	1.18908311804	1.11339863190	1.07464279950
2.46740110027	2.48629840670	2.47176446803	2.46844909132
4.04692529140	4.11437007021	4.06370889386	4.05113570148
9.86960440109	10.1510415461	9.93463029109	9.88528726798
9.86960440109	10.2412894816	9.96756071314	9.89457577220
10.8448542781	11.1417152351	10.9170396386	10.8627059484
12.2648958490	12.9225521063	12.4837921249	12.3632618034
12.3370055014	12.9298573556	12.5018153217	12.3739399986
19.7392088022	20.8736513669	20.0113947868	19.8065536235
21.2441074562	23.2135294482	21.8680311215	21.4675123922

2D cracked square with mixed B.C.: nodal elements

Dirichlet boundary conditions (tangential component) on the exterior boundary and Neumann natural boundary conditions (tangential component of curl) on the crack

Comparison with the results of the corresponding Laplace eigenproblem

Nonuniform mesh

Laplace	64×64	$\|p - p_{LS}\|_0$
1.00526460385	1.02814194204	0.002040
2.46826721357	2.46909837051	0.000155
4.04832244210	4.04988771464	0.001345
4.93711031612	4.93948684715	0.000323
10.8626932383	10.8729891911	0.002266
12.1740192268	12.2224527721	0.007155
12.3502334823	12.3601041072	0.000378
21.0806427612	21.1545298435	0.009691
22.2844191881	22.3239694183	0.007605
23.9235100015	23.9568210108	0.003730

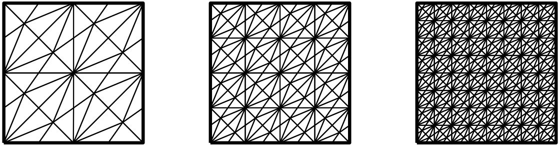
Comparison with other nodal schemes

We consider the following schemes

- Standard nodal elements on Powell–Sabin meshes
  ⟨Wong–Cendes ’88⟩
  ⟨B.–Guzmán–Neilan ’22⟩
  ⟨B.–Gong–Guzmán–Neilan ’22⟩
- Augmented nodal formulation with mesh dependent stabilization ⟨Badia–Codina ’12⟩
  ⟨B.–Codina–Türk in prep.⟩
- A mixed method with nodal elements ⟨Du–Duan ’20⟩
- Nodal elements and control of divergence in fractional norms ⟨Bonito–Guermond ’11⟩

Standard nodal elements on Powell–Sabin meshes

$$\boxed{(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v}}$$



Exact	Computed (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0163	1.0045 (1.9)	1.0011 (2.0)	1.0003 (2.0)	1.0001 (2.0)
1	1.0445	1.0113 (2.0)	1.0028 (2.0)	1.0007 (2.0)	1.0002 (2.0)
2	2.0830	2.0300 (1.5)	2.0079 (1.9)	2.0020 (2.0)	2.0005 (2.0)
4	4.2664	4.1212 (1.1)	4.0315 (1.9)	4.0079 (2.0)	4.0020 (2.0)
4	4.2752	4.1224 (1.2)	4.0316 (2.0)	4.0079 (2.0)	4.0020 (2.0)
5	5.2244	5.1094 (1.0)	5.0326 (1.7)	5.0084 (2.0)	5.0021 (2.0)
5	5.5224	5.2373 (1.1)	5.0647 (1.9)	5.0164 (2.0)	5.0041 (2.0)
8	5.8945	8.3376 (2.6)	8.1198 (1.5)	8.0314 (1.9)	8.0079 (2.0)
9	6.3737	9.5272 (2.3)	9.1498 (1.8)	9.0382 (2.0)	9.0096 (2.0)
9	6.8812	9.5911 (1.8)	9.1654 (1.8)	9.0420 (2.0)	9.0105 (2.0)
zeros	7	39	175	735	3007
DOF	46	190	766	3070	12286

Based on Kikuchi formulation

$\mathcal{X} = \boldsymbol{H}_0(\mathrm{rot}) \times H^1_0(\Omega)$

Find $[\mathbf{u}, p] \in \mathcal{X}$ and $\lambda \in \mathbb{R}$ such that

$$B([\mathbf{u}, p], [\mathbf{v}, q]) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall [\mathbf{v}, q] \in \mathcal{X}$$

with

$$B([\mathbf{u}, p], [\mathbf{v}, q]) = (\mathrm{rot}\,\mathbf{u}, \mathrm{rot}\,\mathbf{v}) + (\mathbf{grad}\, p, \mathbf{v}) - (\mathbf{grad}\, q, \mathbf{u})$$

Orthogonal subscales method

$\mathcal{X}_h \subset \mathcal{X}$

find $[\mathbf{u}_h, p_h] \in \mathcal{X}_h$ and $\lambda_h \in \mathbb{R}$ such that

$$B_S([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall [\mathbf{v}_h, q_h] \in \mathcal{X}_h$$

with

$$\begin{aligned} B_S([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) = {} & B([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) \\ & + \tau_p \sum_K (P^\perp(\nabla p_h), P^\perp(\nabla q_h)) \\ & + \sum_K \tau_{\mathbf{u}} (P^\perp(\nabla \cdot \mathbf{u}_h), P^\perp(\nabla \cdot \mathbf{v}_h))_K \end{aligned}$$

The stabilization parameters are defined as
$\tau_p = \dfrac{L_0^2}{\mu}, \quad \tau_{\mathbf{u}} = c\mu \dfrac{h^2}{L_0^2}$, where $L_0 = \varsigma \ell$, ℓ is a characteristic length, and ς and c are algorithmic constants

The Du–Duan scheme

Based on Powell–Sabin meshes and on the Kikuchi formulation

Find $(\mathbf{u}_h, p_h) \in V_{h/2} \times Q_h$ such that

$$\begin{cases} (\operatorname{rot} \mathbf{u}_h, \operatorname{rot} \mathbf{v}) + (\operatorname{div} \mathbf{v}, p_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}) & \forall \mathbf{v} \in V_{h/2} \\ (\operatorname{div} \mathbf{u}_h, q) = 0 & \forall q \in Q_h \end{cases}$$

The Bonito–Guermond scheme

Main idea: to impose the divergence free constraint with a penalty term in the fractional negative norm of $H^{-\alpha}(\Omega)$, $1/2 < \alpha < 3/2$

Find $\mathbf{u}_h \in V_h$ such that

$$(\operatorname{rot} \mathbf{u}_h, \operatorname{rot} \mathbf{v}) + \langle \operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{v} \rangle_{H^{-\alpha}} = \lambda_h (\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in V_h$$

The scalar product in $H^{-\alpha}$ requires a careful numerical treatment

L-shaped and cracked domain: the comparison

Approximation of the first mode on comparable meshes

L-shaped

$(h = 1/40)$

Exact	Least squares	Powell-Sabin	Badia–Codina
1.47562182408	1.50047387698	1.46453044345	1.77077310272
	Du–Duan	Bonito–Guermond	
	1.46360943009(*)	1.541	

Cracked square

$(h = 1/32)$

Exact	Least squares	Powell-Sabin	Badia–Codina
1.03407400850	1.11339863190	0.99258026882	2.00031262362

Outline

Some introductory examples

Maxwell's eigenvalues and finite element least squares

Joint work with F. Bertrand and L. Gastaldi

Numerical analysis in 2D

Numerical tests in 2D

The 3D case

The 3D case

$$\|\mathbf{v}-\mathbf{curl}\,\mathbf{q}\|_0^2+\|\,\mathbf{curl}\,\mathbf{v}\|_0^2 \geq \alpha\|\mathbf{v}\|_{\mathbf{curl}}^2+\|\mathbf{q}\|_0^2 \qquad \begin{array}{l}\forall \mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \forall \mathbf{q} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}^0)\end{array}$$

Take $V_h \subset \boldsymbol{H}_0(\mathbf{curl})$ and $Q_h \subset \mathbf{Q} = \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}^0)$ and find λ_h and $\mathbf{p}_h \in Q_h$ with $\mathbf{p}_h \neq 0$ such that for some $\mathbf{u}_h \in V_h$

$$\begin{cases} (\mathbf{u}_h, \mathbf{v}) + (\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}) - (\mathbf{curl}\,\mathbf{p}_h, \mathbf{v}) = \lambda_h(\mathbf{p}_h, \mathbf{curl}\,\mathbf{v}) & \forall \mathbf{v} \in V_h \\ -(\mathbf{u}_h, \mathbf{curl}\,\mathbf{q}) + (\mathbf{curl}\,\mathbf{p}_h, \mathbf{curl}\,\mathbf{q}) = 0 & \forall \mathbf{q} \in Q_h \end{cases}$$

Nonconforming approximations

We neglect, or weaken, the inclusion $Q_h \subset \boldsymbol{H}_0(\mathrm{div}^0)$
We consider an edge-edge approximation and a nodal-nodal approximation

The good element (edge-edge)

$\mathcal{N}_k$: Nédélec *edge* finite element space
$\mathcal{L}_{k+1}$: Lagrangian *nodal* finite element space ($k = 0, 1, 2, \dots$)

$V_h = \mathcal{N}_k \cap \boldsymbol{H}_0(\mathbf{curl})$
$Q_h = \{\boldsymbol{\tau} \in \mathcal{N}_k : (\boldsymbol{\tau}, \mathbf{grad}\,\phi) = 0 \quad \forall \phi \in \mathcal{L}_{k+1} \cap L^2_0(\Omega)\}$

Discrete Friedrichs inequality

$$\|\mathbf{q}\|_0 \le C \|\,\mathbf{curl}\,\mathbf{q}\|_0 \qquad \forall \mathbf{q} \in Q_h$$

Then Strang lemma implies uniform convergence of source problem and convergence of the eigenmodes

Edge elements without orthogonality to gradients?

$$\begin{pmatrix} \mathsf{A} & -\mathsf{B}^\top \\ -\mathsf{B} & \mathsf{C} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = \lambda_h \begin{pmatrix} 0 & \mathsf{B}^\top \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}$$

Gradients of nodal elements are in the kernel of both $\mathsf{B}^\top$ and C

$\mathsf{y} \in \ker(\mathsf{C}) \cap \ker(\mathsf{B}^\top)$ implies that for $\mathsf{x} = 0$ our problem degenerates to

$$0 = \lambda_h 0$$

Singular eigenvalue problem

How do solvers behave in presence of degenerate problems?

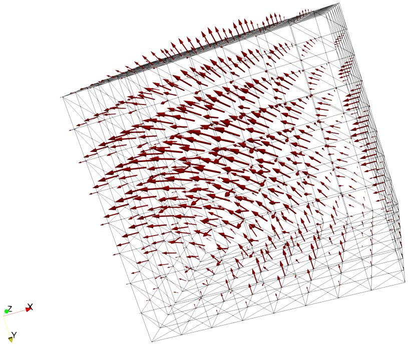
Dealing with the kernel

$8 \times 8 \times 8$ uniform mesh of the cube of size π (dim. $8,368$)

FEniCS, Slepc, Shift and invert (MUMPS), shift=0, first 20 e.v.'s

Exact	Computed
2.	2.07951517
2.	2.09120018
2.	2.09120018
3.	3.20241311
3.	3.20241311
5.	5.41567031
5.	5.41567033
5.	5.42156022
5.	5.59798925
5.	5.61057018
5.	5.61057049
6.	6.61030356
6.	6.61617398
6.	6.61617408
6.	6.70685034
6.	7.05900904
6.	7.05901042
8.	9.34616163
8.	9.35551432
8.	10.52512168

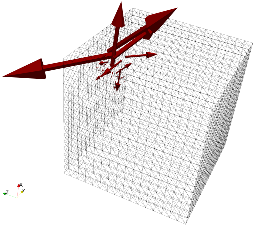
$\mathbf{u}_1$



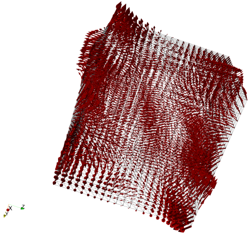
$16 \times 16 \times 16$ uniform mesh of the cube of size π (dim. $62,048$)

Exact	$\Re(\lambda_h)$	$\Im(\lambda_h)$
	0.7803830	0.
2.	2.01975410	0.
2.	2.02244234	0.
2.	2.02244234	0.
3.	3.05012488	0.
3.	3.05012488	0.
	3.0711636	0.11902345
	3.0711636	-0.11902345
	-3.3439379	0.
	-1.2848608	4.75388432
	-1.2848608	-4.75388432
5.	5.10118350	0.
5.	5.10118351	0.
5.	5.10240155	0.
5.	5.14412075	0.
5.	5.14567972	0.
5.	5.14567976	0.
?	5.73760193	0.
?	5.91143439	0.
?	6.15052873	0.

$\mathbf{p}_1$ $(\mathbf{u}_1 = 0)$



$\mathbf{u}_2$



The nodal-nodal element

$$V_h = (\mathcal{L}_{k'})^3 \cap \boldsymbol{H}_0(\mathbf{curl})$$
$$Q_h = (\mathcal{L}_{k''})^3 \cap \boldsymbol{H}(\mathbf{curl})$$

No orthogonality to gradients is imposed

Unless very special meshes are used, no (or very few) gradients are present in the finite element space Q_h

Friedrichs inequality?

In general, Friederichs inequality is not satisfied unless very particular meshes are used (see computations of curl curl eigenvalues with nodal elements)

Question: how do the computations look like?

Uniform mesh

Kernel is present and Friedrichs inequality is satisfied

FEniCS, Slepc, Shift and invert (MUMPS), shift=0, first 20 e.v.'s

$2 \times 2 \times 2$		$4 \times 4 \times 4$		$8 \times 8 \times 8$	
Exact	Computed	Exact	Computed	Exact	Computed
?	-5.84151596e+02	2.	2.48580677	2.	2.11291000
?	-5.84151596e+02	2.	2.48580677	2.	2.11291000
?	-2.26585964e+02	2.	2.49163531	2.	2.11409720
?	-3.76223407e+01	3.	4.02714008	3.	3.24435577
?	-5.91245685e-01	3.	4.02714008	3.	3.24435577
?	-5.91245685e-01	5.	7.50071401	5.	5.52735973
?	4.69343924e+00	5.	7.53369811	5.	5.52969102
?	4.69343924e+00	5.	7.53369811	5.	5.52969102
?	4.73296417e+00	5.	8.77306762	5.	5.77540466
?	8.90567010e+00	5.	9.21255939	5.	5.80254432
?	8.90567010e+00	5.	9.21255939	5.	5.80254432
?	2.96877209e+01	6.	9.29847169	6.	6.74640771
?	2.96877209e+01	6.	9.29847169	6.	6.74640771
?	3.01350325e+01	6.	9.43677471	6.	6.76987621
?	3.25762080e+01	6.	9.94329423	6.	6.82111849
?	5.29331439e+01	6.	13.26657234	6.	7.35145685
?	5.29331439e+01	6.	13.26657235	6.	7.35145686
?	1.50494349e+02	8.	16.80506352	8.	9.77529705
?	1.50494349e+02	8.	16.81874281	8.	9.77529705
?	7.48172598e+02	8.	17.76178205	8.	9.77748225

Uniform mesh (convergence)

$8 \times 8 \times 8$		$16 \times 16 \times 16$		$32 \times 32 \times 32$	
Exact	Computed	Exact	Computed	Exact	Computed
2.	2.11291000	2.	2.02783266	2.	2.00695207
2.	2.11291000	2.	2.02783266	2.	2.00695207
2.	2.11409720	2.	2.02814199	2.	2.00703057
3.	3.24435577	3.	3.06083848	3.	3.01522483
3.	3.24435577	3.	3.06083848	3.	3.01522483
5.	5.52735973	5.	5.12806297	5.	5.03186344
5.	5.52969102	5.	5.12817038	5.	5.03190592
5.	5.52969102	5.	5.12817038	5.	5.03190592
5.	5.77540466	5.	5.18786467	5.	5.04707801
5.	5.80254432	5.	5.18957476	5.	5.04707801
5.	5.80254432	5.	5.18957476	5.	5.04710553
6.	6.74640771	6.	6.19021567	6.	6.04852542
6.	6.74640771	6.	6.19021567	6.	6.04852542
6.	6.76987621	6.	6.19364967	6.	6.04946125
6.	6.82111849	6.	6.19978233	6.	6.04988497
6.	7.35145685	6.	6.31890416	6.	6.07891777
6.	7.35145686	6.	6.31890416	6.	6.07891777
8.	9.77529705	8.	8.43769526	8.	8.10955272
8.	9.77529705	8.	8.43769526	8.	8.10955272
8.	9.77748225	8.	8.43845509	8.	8.10973764

Nonuniform mesh

Kernel is empty or very small and Friedrichs inequality is not satisfied

$8 \times 8 \times 8$		$16 \times 16 \times 16$		$32 \times 32 \times 32$	
Exact	Computed	Exact	Computed	Exact	Computed
2.	2.37376889	2.	2.08572641	2.	2.02123000
2.	2.37596566	2.	2.08655713	2.	2.02135377
2.	2.38729739	2.	2.08680079	2.	2.02139551
3.	3.93970069	3.	3.19564527	3.	3.04803275
3.	3.99751371	3.	3.20009678	3.	3.04847792
5.	7.83683485	5.	5.55442161	5.	5.13643073
5.	7.88044188	5.	5.56397328	5.	5.13666711
5.	7.94628286	5.	5.56789994	5.	5.13690603
5.	8.01359808	5.	5.57526288	5.	5.13754924
5.	8.06695053	5.	5.58076755	5.	5.13812634
5.	8.26396691	5.	5.58299225	5.	5.13861874
6.	10.02558981	6.	6.80197758	6.	6.19728454
6.	10.13590133	6.	6.80766694	6.	6.19788901
6.	10.19899077	6.	6.81452833	6.	6.19817182
6.	10.30561540	6.	6.83462623	6.	6.19861794
6.	10.43259596	6.	6.83915509	6.	6.19889002
6.	10.49346686	6.	6.84227011	6.	8.35158586
8.	16.38805084	8.	9.47802503	8.	8.35249384
8.	16.53924802	8.	9.49627505	8.	8.35541674
8.	17.04917019	8.	9.51225702	8.	8.35541674

Wrap-up

2D

All conforming dicretizations give optimal results

3D

edge-edge: good results when taking into account orthogonality to gradients (theory OK)
Otherwise troubles when the kernel is too large

nodal-nodal: troubles when the kernel is too small on uniform meshes, otherwise good results
Good results on non uniform meshes despite the lack of theoretical results

Conclusions

- Edge elements provide the most natural choice for the approximation of the curl curl problem (de Rham complex, etc.)
- Big kernel can be eliminated at algebraic level or by the use of Lagrange multiplier
- Nodal elements are sometimes appealing and it is an interesting problem to see if they can work
- Two dimensional schemes based on nodal elements are effective and theoretically studied (special mesh sequence for standard formulation or any mesh and least-squares based formulation). Some schemes can be extended to 3D
- Least squares: 2D singular problems and three dimensional schemes are less straightforward to analyze; numerical results seem to be promising