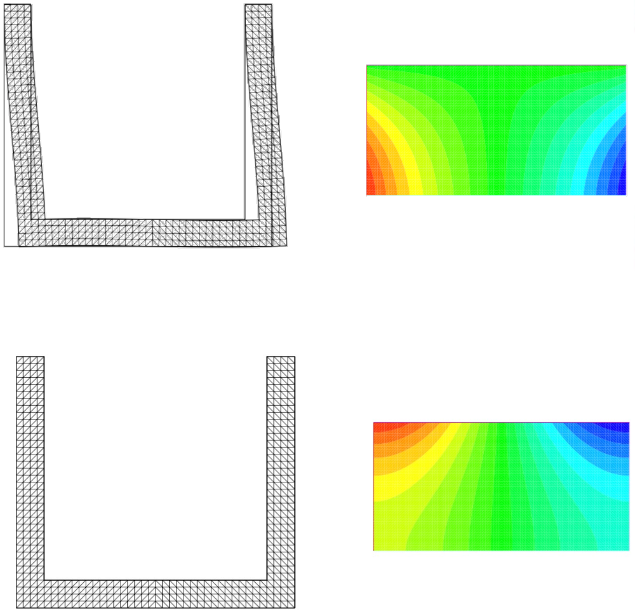
Approximation of eigenvalue problems with MINRES

Fleurianne Bertrand

Mathematics of Computational Science

UNIVERSITY OF TWENTE.

Motivation: eigenvalues from coupled differential equations



Bermúdez/Durán/Rodríguez 1998
Meddahi/Mora/Rodríguez 2014
Brenner/Çeşmelioğlu/Cui/Sung 2019

Abstract Framework: $\mathcal{L}_i u_i = \lambda u_i$ in Ω $\quad g_j(u_1, \ldots, u_n) = 0$ on Γ_j

$\sigma_i = \mathcal{C}_i(u_i)$, div $(\mathcal{A}_i \sigma_i) = \lambda u_i$

Key objectives for the discretisations

1. Robust discretisations of the corresponding source formulation
   $\Rightarrow$ Discretise dual variables σ_i
2. Flexibility
   $\Rightarrow$ Circumvent inf-sup/LBB condition
3. Allows for efficient solving strategies (avoid eigenvalue crossing)
   $\Rightarrow$ Inherent error estimator

$\Rightarrow$ **Relevance of discretisation method in taylored solution strategy**

ROM for parametric eigenvalue problems

Goal

Replicate input-output behaviour of large-scale system over a certain range of parameter inputs

Large-Scale Model

$$\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{u}(\boldsymbol{\mu}) = \boldsymbol{F}(\boldsymbol{\mu})$$
$$\boldsymbol{s}(\boldsymbol{\mu}) = \underline{\boldsymbol{L}}(\boldsymbol{\mu})^{\boldsymbol{T}}\underline{\boldsymbol{u}}(\boldsymbol{\mu})$$

Outputs of Interest $s(\mu)$

Parameter Inputs μ

Reduced-Order Model

$$\boldsymbol{A}_N(\boldsymbol{\mu})\underline{u}_N(\boldsymbol{\mu}) = \underline{F}_N(\boldsymbol{\mu})$$
$$\boldsymbol{s}_N(\boldsymbol{\mu}) = \underline{L}_N(\boldsymbol{\mu})^{T u_N(\boldsymbol{\mu})}$$

Outputs of Interest $s_N(\mu)$

Parameter Inputs μ

dim(Large-Scale Model) >>dim(Reduced-Order Model)=N

The approximation error is small if $\|\underline{\boldsymbol{u}}(\boldsymbol{\mu}) - \underline{\boldsymbol{u}}_{\boldsymbol{N}}(\boldsymbol{\mu})\| \leq \varepsilon$ and $|\mathbf{s}(\boldsymbol{\mu}) - \mathbf{s}_N(\mu)| \leq \tilde{\varepsilon}\ \forall \mu \in \mathcal{D}$.

ROM for parametric eigenvalue problems

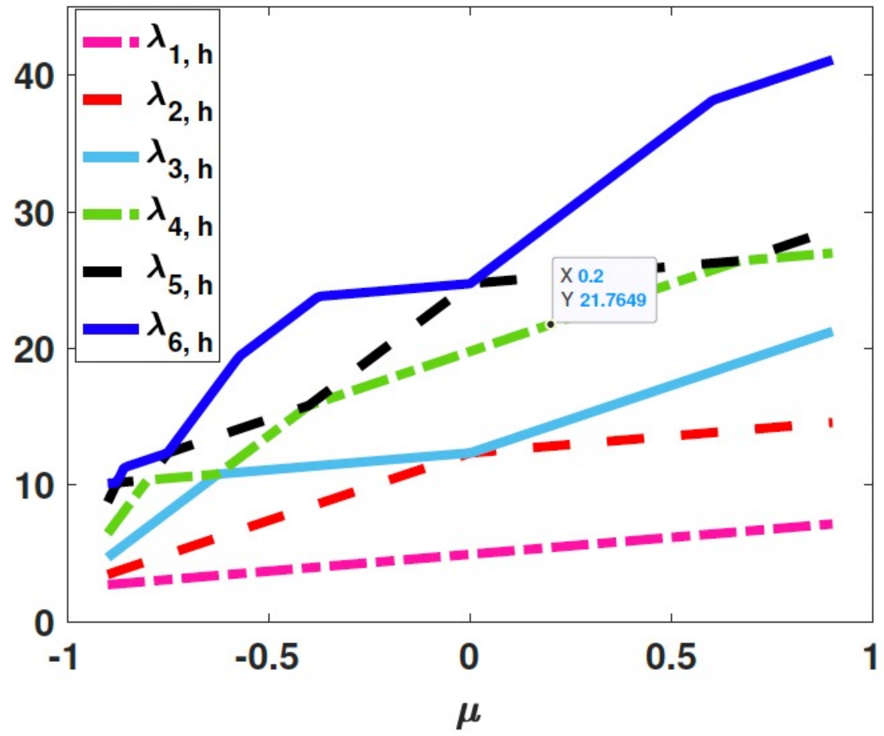
- Given a window of values $[\lambda_{min}, \lambda_{max}] \subset \mathbb{R}_+$. for each $\mu \in \mathcal{M}$, find eigenvalues $\lambda(\mu) \in [\lambda_{min}, \lambda_{max}]$ and non-vanishing eigenfunctions $u(\mu) \in V$ such that, for all $v \in V$ it holds

$$\begin{cases} -\operatorname{div}(A(\mu)\nabla u(\mu)) = \lambda(\mu)u(\mu) & \text{in } \Omega = (0,1)^2 \\ u(\mu) = 0 & \text{on } \partial\Omega \end{cases}$$

- Simple test with diffusion $A(\mu) \in \mathbb{R}^{2\times 2}$ given by the diagonal matrix

$$A(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\mu \end{pmatrix}.$$

- Eigenfunctions independent of μ



(b) First six eigenvalues $\lambda_{1,h}, \ldots, \lambda_{6,h}$ computed by the FEM and sorted according to their magnitutde, for $\mu \in \mathcal{M}$.

Motivation: eigenvalues from coupled differential equations

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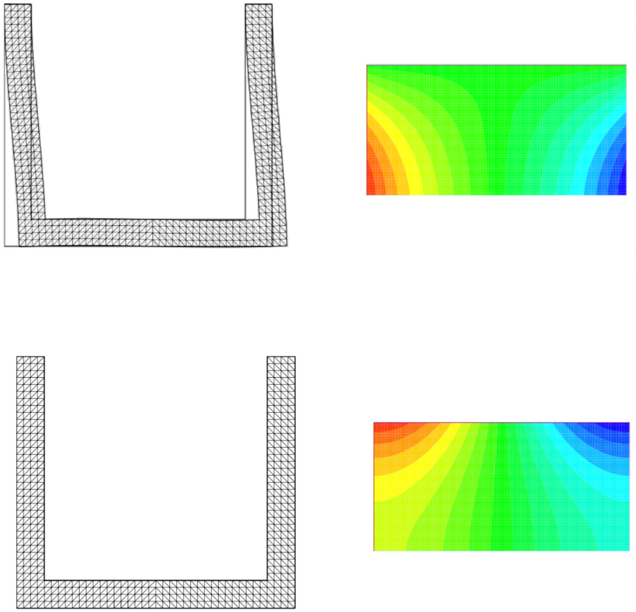

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Outline

- Laplace eigenvalue approximation with LSFEM
- Laplace eigenvalue approximation with DPG
- Vibration of elastic structures
- Outlook and conclusions

FB, Daniele Boffi, First order least-squares formulations for eigenvalue problems, IMAJNA, 2021

FB, Daniele Boffi, Least-Squares Formulations for eigenvalue problems associated with linear elasticity, CAMWA 2020

From source problems to eigenvalue problems

$H, V, V_h \subset V$ Hilbert spaces.

Study of the eigenvalue problem

find $\lambda \in \mathbb{R}, u \in V, u \neq 0:$
$$a(u,v) = \lambda b(u,v) \quad \forall v \in V$$

find $\lambda_h \in \mathbb{R}, u_h \in V_h, u \neq 0:$
$$a(u_h,v_h) = \lambda_h b(u_h,v_h) \quad \forall v_h \in V_h$$

Convergence from Babuska-Osborn theory

Study of the source problem

find $u \in V$
$$a(u,v) = b(f,v) \quad \forall v \in V$$

find $u_h \in V_h$
$$a(u_h,v_h) = b(f,v_h) \quad \forall v_h \in V_h$$

Solution operator: $T : H \to H,\ a(Tf,v) = b(f,v)\ \forall v \in V$
Discrete solution operator: $T_h : H \to H,\ a(T_h f, v_h) = b(f,v_h)\ \forall v_h \in V_h$

If T is compact and self-adjoint, and $\|T - T_h\|_{\mathcal{L}(H,H)} \to 0$ holds, then there is no spurious modes.

Laplace eigenvalue approximation with LSFEM

- find $\lambda \in \mathbb{R}$ and u non vanishing such that $\quad -\Delta u = \lambda u$ in $\Omega \quad$ and $\quad u = 0$ on $\partial\Omega$

- Standard first order formulation: find $\lambda \in \mathbb{R}$ and u non vanishing such that for some $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} - \nabla u = 0 \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{\sigma} = -\lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

- Least squares formulation for the source problem

$$\mathcal{F}(\boldsymbol{\tau}, v) = \|\boldsymbol{\tau} - \nabla v\|^2 + \|\operatorname{div} \boldsymbol{\tau} + f\|^2$$

- Source problem, variational formulation: find $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega)$ and $u \in H_0^1(\Omega)$ such that

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\sigma}, \operatorname{div} \boldsymbol{\tau}) - (\nabla u, \boldsymbol{\tau}) = -(f, \operatorname{div} \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \\ -(\boldsymbol{\sigma}, \nabla v) + (\nabla u, \nabla v) = 0 & \forall v \in H_0^1(\Omega) \end{cases}$$

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- Eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u \in H^1_0(\Omega)$ with $u \neq 0$ such that for some $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega)$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\sigma}, \operatorname{div} \boldsymbol{\tau}) - (\nabla u, \boldsymbol{\tau}) = -\lambda(u, \operatorname{div} \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \\ -(\boldsymbol{\sigma}, \nabla v) + (\nabla u, \nabla v) = 0 & \forall v \in H^1_0(\Omega) \end{cases}$$

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- three families of eigenvalues
  - $\lambda \in \mathbb{R}^+$ corresponding to dim(rank(B))
  - $\lambda = +\infty$ corresponding to $\dim\left(\ker\left(B^\top\right)\right)$
  - $\lambda = +\infty$ corresponding to $\dim\left(\Sigma_h\right)$

$$\begin{pmatrix} A & B^\top \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Since $D = -B^\top$, symmetric and positive definite formulation:

$$Ax = (\lambda + 1) B^\top C^{-1} Bx,$$

$\Rightarrow$ easy computation of the eigenvalues of the first family

Laplace eigenvalue approximation with LSFEM

- $T_F : L^2(\Omega) \to L^2(\Omega)$ such that for $f \in L^2(\Omega)$ $T_F f \in H_0^1(\Omega)$ is the second component of the solution of the variational formulation, i.e. for some $\boldsymbol{\sigma} \in H(\mathrm{div}, \Omega)$:

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\, \boldsymbol{\sigma}, \mathrm{div}\, \boldsymbol{\tau}) - (\nabla T_F f, \boldsymbol{\tau}) = -(f, \mathrm{div}\, \boldsymbol{\tau}) \\ -(\boldsymbol{\sigma}, \nabla v) + (\nabla T_F f, \nabla v) = 0 \end{cases}$$

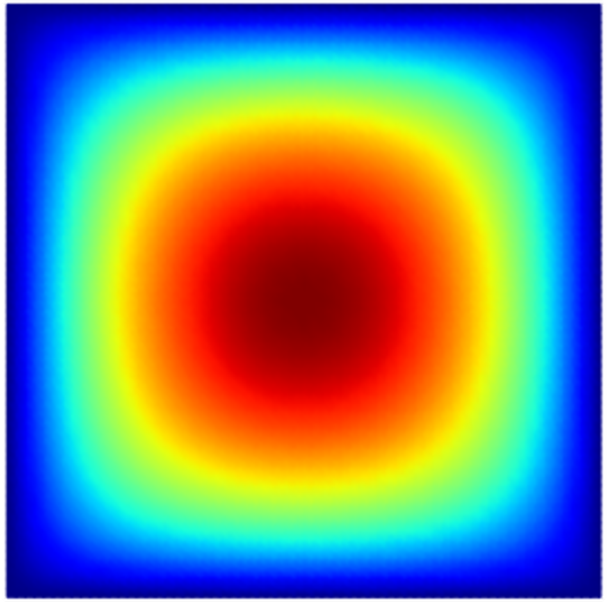
$$\text{for all } \boldsymbol{\tau} \in H(\mathrm{div}, \Omega), \ v \in H_0^1(\Omega)$$

- T_F is compact and self-adjoint.
- Reciprocals of its non-vanishing eigenvalues in increasing order (tending to $+\infty$):

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$$

The corresponding eigenfunctions are denoted by $\{u_i\}$, $i = 1, 2, \ldots, i, \ldots$.

f=1



Laplace eigenvalue approximation with LSFEM

- $T_{F,h} : L^2(\Omega) \to L^2(\Omega)$ such that given $f \in L^2(\Omega)$, $T_{F,h}f \in U_h$ is the second component of the solution of the Galerkin approximation:

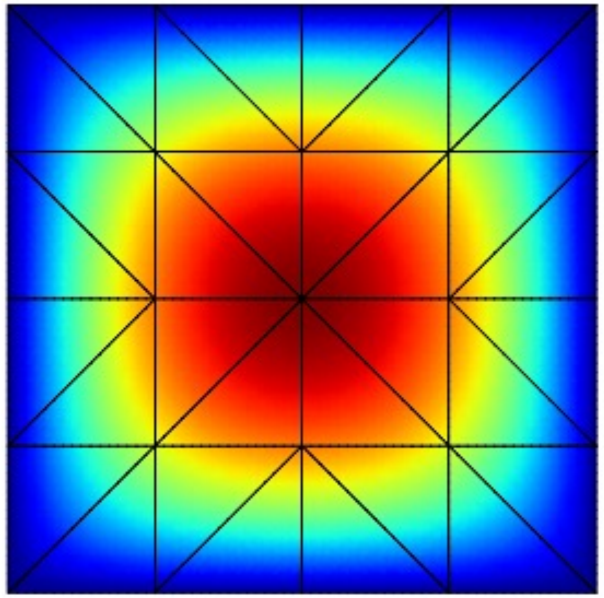
$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\sigma}_h, \operatorname{div} \boldsymbol{\tau}) - (\nabla T_{F,h} f, \boldsymbol{\tau}) = -(f, \operatorname{div} \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \Sigma_h \\ -(\boldsymbol{\sigma}_h, \nabla v) + (\nabla T_{F,h} f, \nabla v) = 0 & \forall v \in U_h \end{cases}$$

- $T_{F,h}$ is compact and self-adjoint.
- reciprocals of its non-vaninshing eigenvalues:

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{i,h} \leq \cdots \leq \lambda_{N(h),h}$$

- The corresponding eigenfunctions are denoted by $\{u_{i,h}\}$
- $N(h) \leq \dim(U_h)$ is the rank of the matrix D $i = 1, 2, \ldots, N(h)$, with the same convention for normalization and multiple eigenvalues.

f=1



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We want $\forall f \in L^2(\Omega)$

$$\|T_F f - T_{F,h}f\|_0 \leq \rho(h)\|f\|_0$$

with $T_F f = u$ and $T_{F,h}f = u_h$

Energy estimates don't imply uniform convergence

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div}} + \|u - u_h\|_1 \leq C \inf_{\substack{\boldsymbol{\tau}_h \in \Sigma_h \\ v_h \in U_h}} \left(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{div}} + \|u - v_h\|_1\right)$$

We want

$$\|T_F f - T_{F,h} f\|_0 \leq \rho(h) \|f\|_0 \quad \forall f \in L^2(\Omega)$$

with $T_F f = u$ and $T_{F,h} f = u_h$

Lack of compactness

$$\operatorname{div} \boldsymbol{\sigma} = f$$

doesn't allow any rate of convergence out of

$\inf_{\boldsymbol{\tau}_h \in \Sigma_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{div}}$

We need L2 estimates

We want to estimate $\|u - u_h\|_0$
Consider the dual problem (related to LL*)
Find $\boldsymbol{\chi} \in H(\mathrm{div}, \Omega)$ and $p \in H_0^1(\Omega)$ such that

$$\begin{cases} (\boldsymbol{\chi}, \boldsymbol{\xi}) + (\mathrm{div}\, \boldsymbol{\chi}, \mathrm{div}\, \boldsymbol{\xi}) - (\nabla p, \boldsymbol{\xi}) = 0 & \forall \boldsymbol{\xi} \in H(\mathrm{div}, \Omega) \\ -(\boldsymbol{\chi}, \nabla q) + (\nabla p, \nabla q) = (u - u_h, q) & \forall q \in H_0^1(\Omega) \end{cases}$$

Consider minimal regularity for polytopal domains
$u \in H^{1+s}(\Omega)$, for some $s > 1/2$
We have
$\boldsymbol{\chi} = \nabla(p + g)$, $\Delta g = u - u_h$, $\Delta p = g - u + u_h$, $\mathrm{div}\, \boldsymbol{\chi} = g$
and

$$\|p\|_{1+s} + \|\boldsymbol{\chi}\|_s + \|\mathrm{div}\, \boldsymbol{\chi}\|_{1+s} \leq C\|u - u_h\|_0$$

We need L2 estimates

Take $\boldsymbol{\xi} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $q = u - u_h$
Use error equation

$$
\begin{aligned}
\|u - u_h\|_0^2 &= (\boldsymbol{\chi}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + (\operatorname{div} \boldsymbol{\chi}, \operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)) - (\nabla p, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\
&\quad - (\boldsymbol{\chi}, \nabla(u - u_h)) + (\nabla p, \nabla(u - u_h)) \\
&= (\boldsymbol{\chi} - \boldsymbol{\tau}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + (\operatorname{div}(\boldsymbol{\chi} - \boldsymbol{\tau}_h), \operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)) \\
&\quad - (\nabla(p - v_h), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\
&\quad - (\boldsymbol{\chi} - \boldsymbol{\tau}_h, \nabla(u - u_h)) + (\nabla(p - v_h), \nabla(u - u_h))
\end{aligned}
$$

We get (for all $\boldsymbol{\tau}_h$ and v_h)

$$
\|u - u_h\|_0^2 \leq C(\|\boldsymbol{\chi} - \boldsymbol{\tau}_h\|_{\operatorname{div}} + \|p - v_h\|_1)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\operatorname{div}} + \|u - u_h\|_1)
$$

Uniform convergence

Theorem

If the finite element spaces satisfy the approximation properties

$$\inf_{\boldsymbol{\tau}_h \in \Sigma_h} \|\boldsymbol{\chi} - \boldsymbol{\tau}_h\|_{\mathrm{div}} \le C h^s (\|\boldsymbol{\chi}\|_s + \|\operatorname{div} \boldsymbol{\chi}\|_s)$$

$$\inf_{v_h \in U_h} \|p - v_h\|_1 \le C h^s \|p\|_{1+s}$$

then the following uniform convergence holds true

$$\|(T_F - T_{F,h}) f\|_0 \le h^s \|f\|_0$$

$\Rightarrow$ The eigensolutions converge and there is no spurious mode

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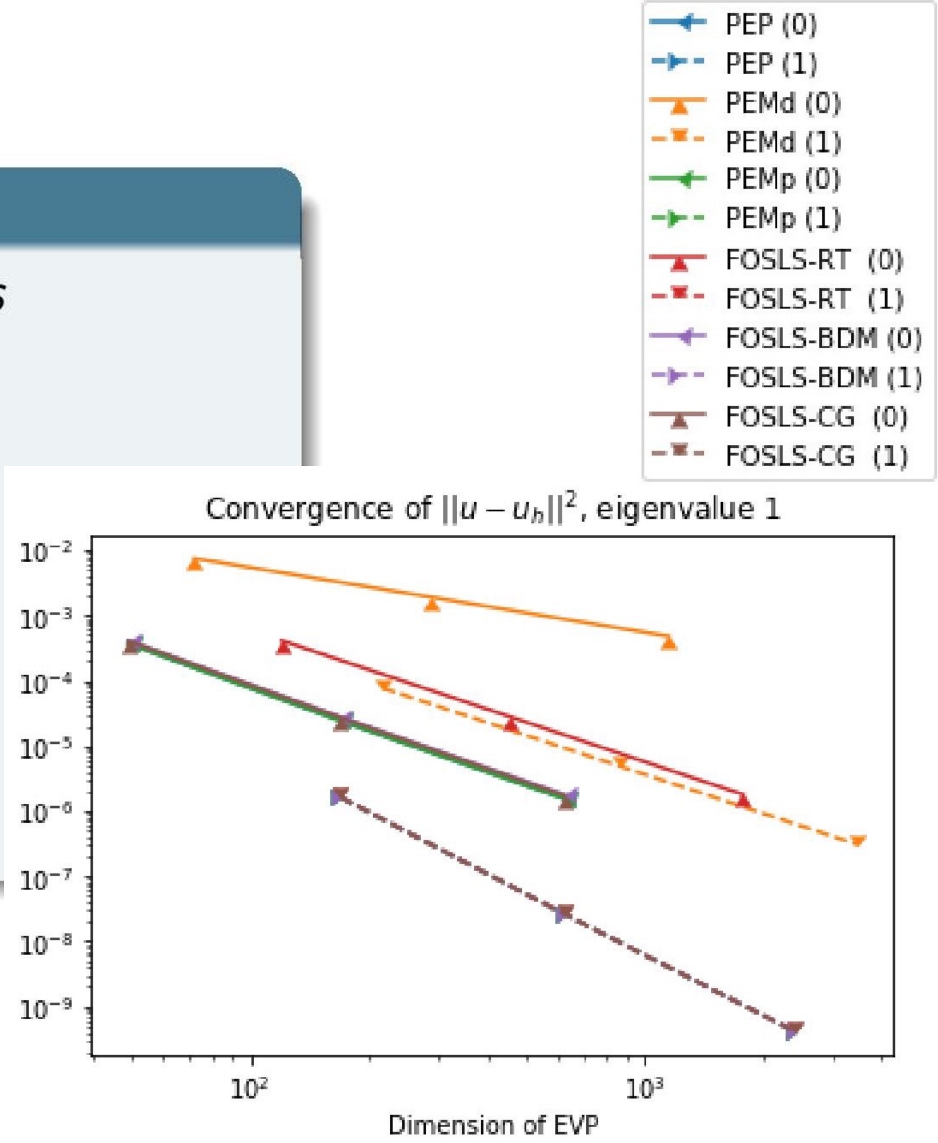

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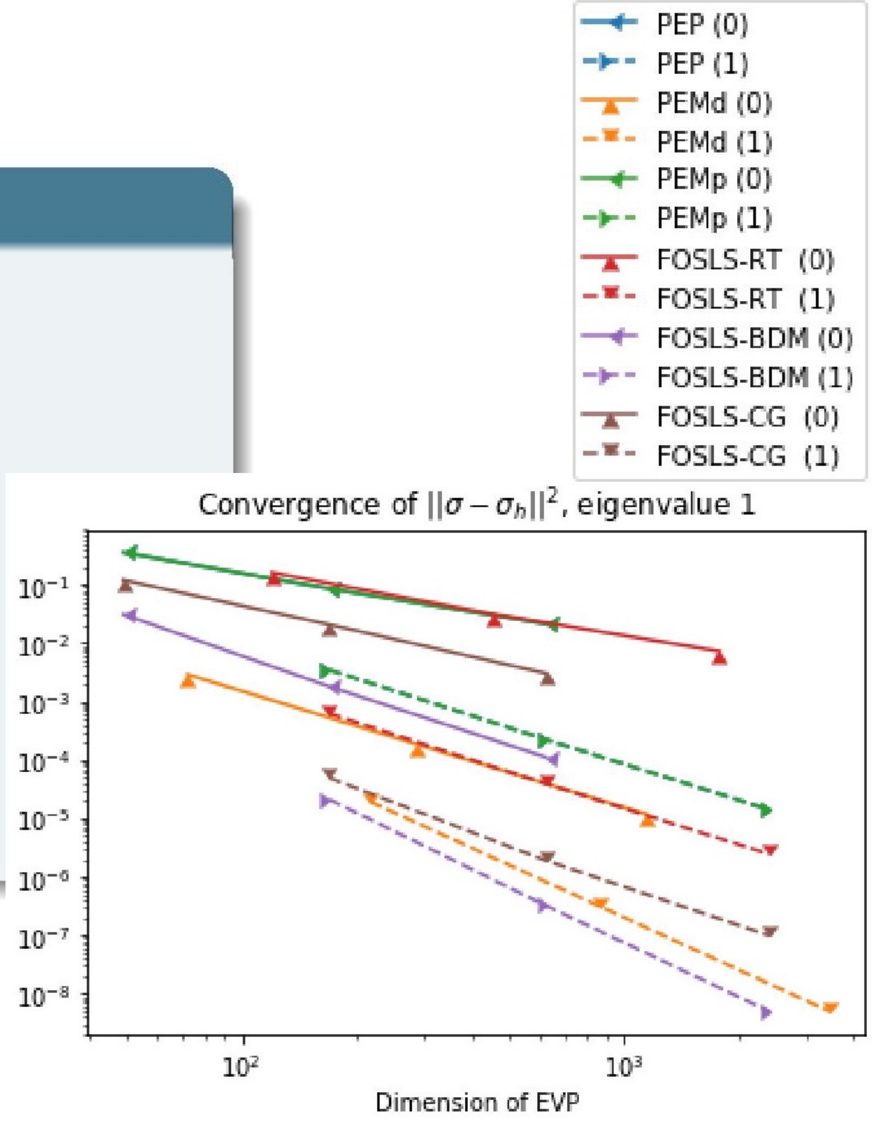

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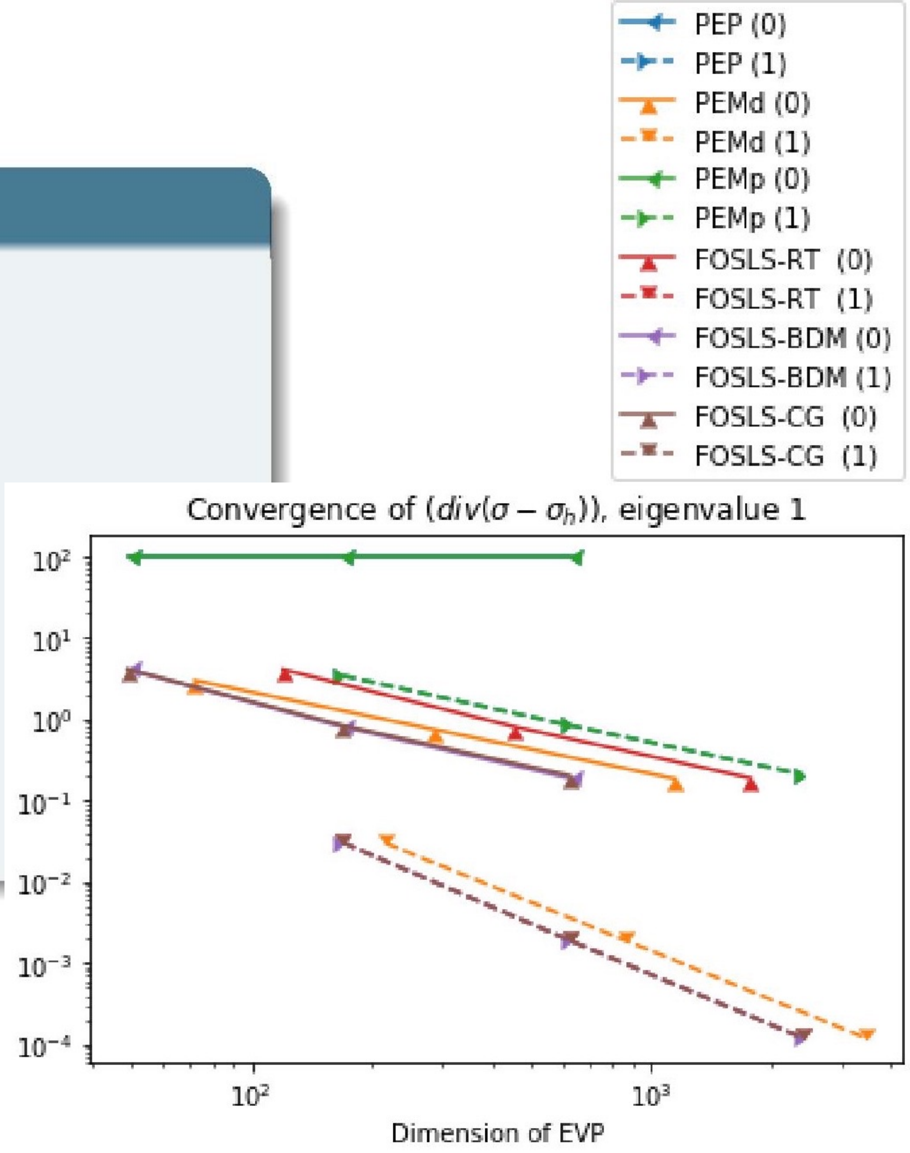


Convergence of the eigenvalues

Theorem (Babŭska–Osborn)

If $\|(T_F - T_{F,h})f\|_{\mathcal{H}} \leq \rho(h)\|f\|_{\mathcal{H}} \quad \forall f \in \mathcal{H}$

$$\hat{\delta}(E_h, E) \leq C\gamma_h, \ \textit{and} \ \max_{j=1,\dots,m} |\lambda_i - \lambda_{i_j,h}| \leq C(\gamma_h \gamma_h^*)^{1/\alpha}.$$

Moreover

$$\delta(E, E_h) \to 0 \quad \text{as } h \to 0 \text{ and } \delta(E, E_h) \leq C\|(T_F - T_{F,h})|_E\|_{\mathcal{L}(H^1)}$$

with

- δ denote as usual the gap between Hilbert subspaces
- E is the continuous eigenspace spanned by $\{u_i, \dots, u_{i+m-1}\}$
- E_h is its discrete counterpart spanned by $\{u_{i,h}, \dots, u_{i+m-1,h}\}$.

Dual Problem

- dual problem: find $\lambda \in \mathbb{R}$ and $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega)$ with $\boldsymbol{\sigma} \neq \mathbf{0}$ such that for some $u \in H_0^1(\Omega)$ it holds

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\sigma}, \operatorname{div} \boldsymbol{\tau}) - (\nabla u, \boldsymbol{\tau}) = 0 & \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \\ -(\boldsymbol{\sigma}, \nabla v) + (\nabla u, \nabla v) = -\lambda(\operatorname{div} \boldsymbol{\sigma}, v) & \forall v \in H_0^1(\Omega) \end{cases}$$

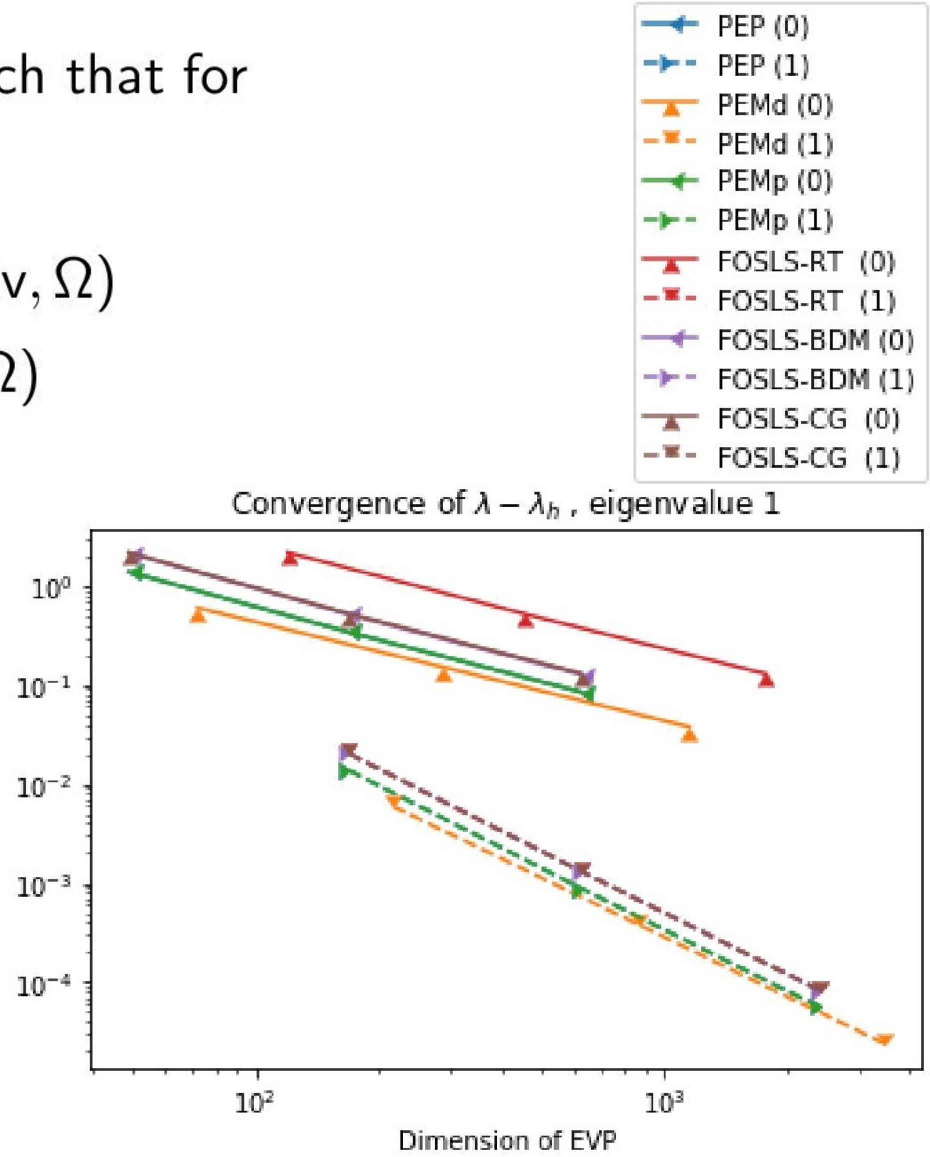
- corresponding matrix form

$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & \mathsf{C} \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ \mathsf{D}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}$$

- corresponding reduced *symmetric* form

$$\mathsf{C}\mathsf{y} = (\lambda + 1)\mathsf{B}\mathsf{A}^{-1}\mathsf{B}^\top \mathsf{y}$$

- Primal and dual problems are equivalent



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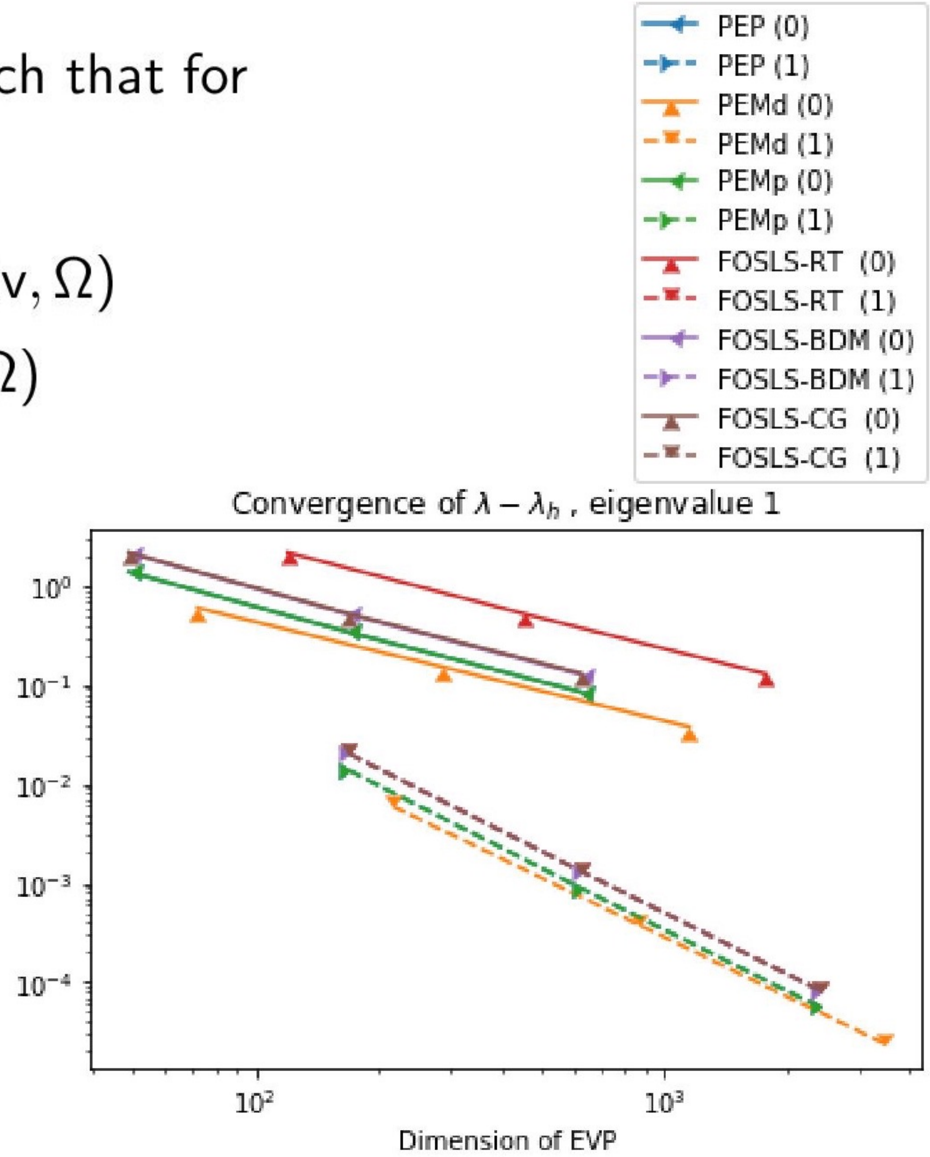


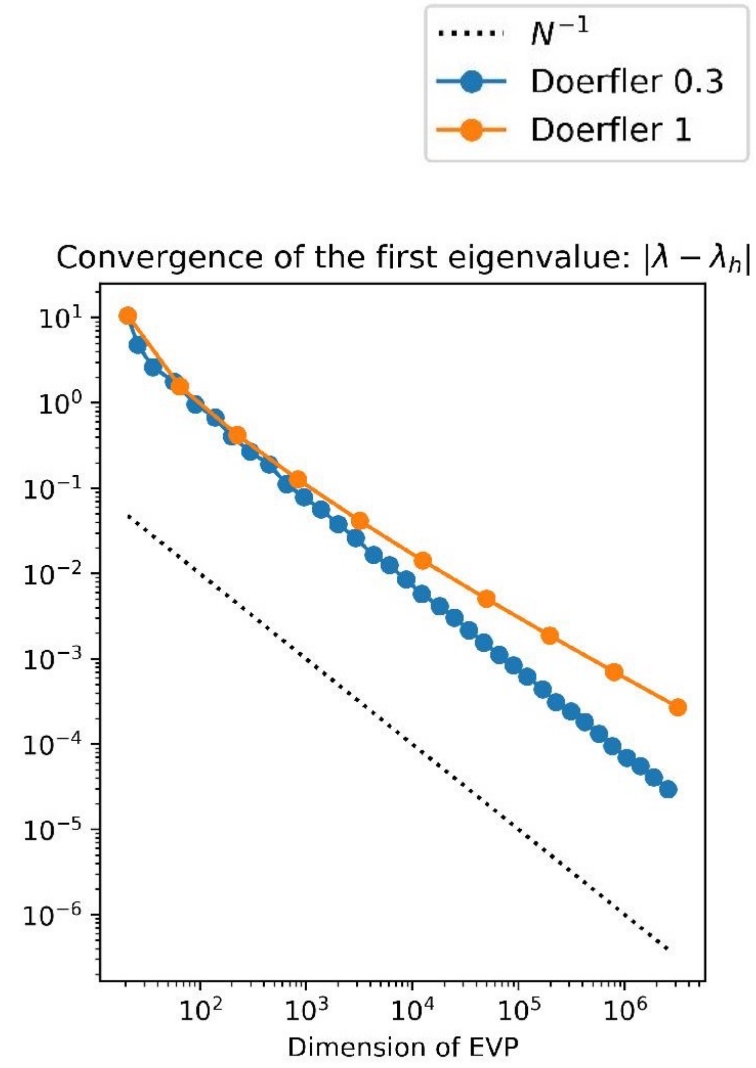
A posteriori

We consider the following error estimator

$$\eta_T^2 = h_T^2 \| \operatorname{div} \boldsymbol{\sigma}_h - \Delta u_h \|_{L^2(T)}^2 + h_T^2 \| \mathrm{curl} \boldsymbol{\sigma}_h \|_{L^2(T)}^2 + \sum_{e \in \partial T} h_e \left(\| [\![\boldsymbol{\sigma}_h \cdot \mathbf{t}]\!] \|_{L^2(e)}^2 + \| [\![\nabla u_h \cdot \mathbf{n}]\!] \|_{L^2(e)}^2 \right)$$

which gives as usual the global estimator

$$\eta_h^2 = \sum_T \eta_T^2$$



Theorem

Reliability (up to higher order term) and efficiency

The curl formulation

- Enriching the formulation with curl σ

$$\mathcal{F}(\boldsymbol{\tau}, v) = \|\boldsymbol{\tau} - \nabla v\|^2 + \|\mathrm{curl}\boldsymbol{\sigma}\|^2 + \|\operatorname{div}\boldsymbol{\tau} + f\|^2$$

- find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ with $u \neq 0$ such that for some $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega) \cap \mathbf{H}(\mathrm{curl}; \Omega)$ it holds

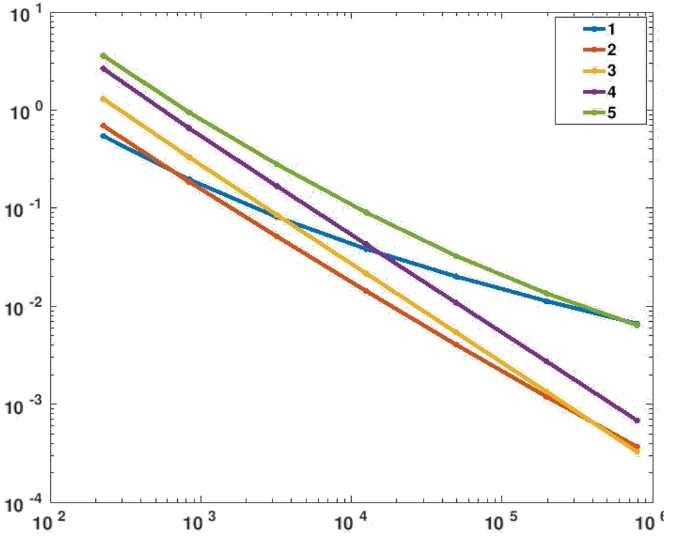
$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau}) + (\mathrm{curl}\boldsymbol{\sigma}, \mathrm{curl}\boldsymbol{\tau}) - (\nabla u, \boldsymbol{\tau}) = -\lambda(u, \operatorname{div}\boldsymbol{\tau}) \\ -(\boldsymbol{\sigma}, \nabla v) + (\nabla u, \nabla v) = 0 \end{cases}$$

$\forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \cap \mathbf{H}(\mathrm{curl}; \Omega), v \in H_0^1(\Omega)$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau}) + (\mathrm{curl}\boldsymbol{\sigma}, \mathrm{curl}\boldsymbol{\tau}) - (\nabla u, \boldsymbol{\tau}) = 0 \\ -(\boldsymbol{\sigma}, \nabla v) + (\nabla u, \nabla v) = -\lambda(\operatorname{div}\boldsymbol{\sigma}, v) \end{cases}$$

$\forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \cap \mathbf{H}(\mathrm{curl}; \Omega), v \in H_0^1(\Omega)$

Approximation of the i-th eigenvalue



M. Costabel. A coercive bilinear form for Maxwell's equations. J. Math. Anal. Appl. 157(2) 1991, Pages 527-541

M. Costabel, M. Dauge. Maxwell and Lamé Eigenvalues on Polyhedra. Math. Methods Appl. Sci., 22(2-4), 1999.

Outline

- Laplace eigenvalue approximation with LSFEM
- Laplace eigenvalue approximation with DPG
- Vibration of elastic structures
- Outlook and conclusions

DPG eigenvalue approximation

- Ideal setting with $U_h \subset U$ and $V_{opt} = T(U_h)$: find $u_h \in U_h$ such that

$$b(u_h, v) = \ell(v) \quad \forall v \in V_{opt}.$$

- Practical setting with $V_h \subset V$, discrete trial-to-test operator T_h and $V_{opt,h} = T_h(U_h)$. Define $\Pi : V \to V_h$ such that for all $u_h \in U_h$ and all $v \in V$

$$b(u_h, v - \Pi v) = 0, \qquad \|\Pi v\|_V \leq C_\Pi \|v\|_V.$$

- Find $u_h \in U_h$ and $\varepsilon_h \in V_h$ such that

$$\begin{cases} (\varepsilon_h, v_h)_V + b(u_h, v_h) = \ell(v_h) & \forall v_h \in V_h \\ \overline{b(z_h, \varepsilon_h)} = 0 & \forall z_h \in U_h. \end{cases}$$

DPG eigenvalue approximation

- $U = U_0 \times U_1$, where U_0 is a functional space defined on Ω
- Hilbert pivot space $\mathcal{H}$ with $U_0 \subset \mathcal{H} \simeq \mathcal{H}' \subset U_0'$
- bilinear form $m : \mathcal{H} \times V \to \mathbb{C}$

Eigenvalue Problem

Find $\lambda \in \mathbb{C}$ and $u = (u_0, u_1) \in U = U_0 \times U_1$ with $u_0 \neq 0$ such that

$$b(u, v) = \lambda m(u_0, v) \quad \forall v \in V.$$

Corresponding solution operator: $T_F : \mathcal{H} \to \mathcal{H}$ such that $T_F f \in \mathcal{H}$ is the component u_0 of the solution $u \in U$ to

$$b(u, v) = m(f, v) \quad \forall v \in V.$$

DPG eigenvalue approximation

Consider $U_{0,h} \subset U_0$ and $U_{1,h} \subset U_1$.

Discrete eigenvalue Problem

Find $\lambda_h \in \mathbb{C}$ such that for some $u_h = (u_{0,h}, u_{1,h}) \in U_h = U_{0,h} \times U_{1,h}$ with $u_{0,h} \neq 0$ and some $\varepsilon_h \in V_h$ it holds

$$\begin{cases} (\varepsilon_h, v_h)_V + b(u_h, v_h) = \lambda_h m(u_{0,h}, v_h) & \forall v_h \in V_h \\ \overline{b(z_h, \varepsilon_h)} = 0 & \forall z_h \in U_h. \end{cases}$$

Matrix form:

$$\begin{pmatrix} A & B^\top \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

DPG eigenvalue approximation

discrete counterpart $T_{F,h} : \mathcal{H} \to \mathcal{H}$ of T_F

$T_{F,h} f \in U_{0,h} \subset \mathcal{H}$ is the component $u_{0,h} \in U_{0,h}$ of the solution $u_h \in U$ of the following problem, for some $\varepsilon_h \in V_h$,

$$\begin{cases} (\varepsilon_h, v_h)_V + b(u_h, v_h) = m(f, v_h) & \forall v_h \in V_h \\ \overline{b(z_h, \varepsilon_h)} = 0 & \forall z_h \in U_h. \end{cases}$$

Theorem (Babŭska–Osborn)

If

$$\|(T_F - T_{F,h})f\|_{\mathcal{H}} \leq \rho(h)\|f\|_{\mathcal{H}} \quad \forall f \in \mathcal{H}$$

with $\rho(h) \to 0$ as $h \to 0$, then the discrete eigenvalues and eigenfunctions converge to the continuous ones.

DPG eigenvalue approximation (primal formulation)

- $U_0 = H_0^1(\Omega),\ U_1 = H^{-1/2}(\partial\Omega_h),\ \mathcal{H} = H^1(\Omega),\ V = H^1(\Omega_h)$
  $b(u,\hat\sigma_n; v) = (\nabla u, \nabla v)_{\Omega_h} - \langle \hat\sigma_n, v\rangle_{\partial\Omega_h}\ \ m(u,v) = (u,v)_{\Omega_h},$

- for $k \geq 1$: $U_{h,0} = S_0^k(\Omega_h) := P_k(\Omega_h) \cap C(\bar\Omega)$
  $U_{h,1} = P_{k-1}(\partial\Omega_h) \cap U_1,\ V_h = P_{k+1}(\Omega_h).$

- D+G 2013:
  $$\|u - u_h\|_{H^1(\Omega)} + \|\hat\sigma_n - \hat\sigma_{h,n}\|_{H^{-1/2}(\partial\Omega_h)} \leq C \inf_{(w_h,\hat r_{h,n})\in U_h}(\|u - w_h\|_{H^1(\Omega)} + \|\hat\sigma_n - \hat r_{h,n}\|_{H^{-1/2}(\partial\Omega_h)}).$$

Uniform convergence, primal case

Let u belongs to $H^{1+s}(\Omega)$ for $s \in (1/2, k+1]$. Then,

$$\|(T_F - T_{F,h})f\|_{H^1(\Omega)} \leq Ch^s\|f\|_{H^1(\Omega)}.$$

DPG eigenvalue approximation (primal formulation)

adjoint problem, see

T. Bouma, J. Gopalakrishnan, and A. Harb, Convergence rates of the DPG method with reduced test space degree, Comput. Math. Appl. 68 (2014)

given $g \in \mathcal{H}$, find $\varepsilon^* \in V$ and $\mathbf{u}^* = (u^*, \hat{\sigma}_n^*) \in U$ such that

$$\begin{cases} (\varepsilon^*, w)_V + (\nabla u^*, \nabla w)_{\Omega_h} - \langle \hat{\sigma}_n^*, w \rangle_{\partial\Omega_h} = 0 & \forall w \in V \\ (\nabla \varepsilon^*, \nabla v)_{\Omega_h} = (g, v)_{\Omega_h} & \forall v \in U_0 \\ \langle \hat{\tau}_n, \varepsilon^* \rangle_{\partial\Omega_h} = 0 & \forall \hat{\tau}_n \in U_1 \end{cases}$$

quasi-optimal a priori estimate:

$$\begin{aligned} &\|\varepsilon^* - \varepsilon_h^*\|_V + \|u^* - u_h^*\|_{U_0} + \|\hat{\sigma}_n^* - \hat{\sigma}_{n,h}^*\|_{U_1} \\ &\quad \leq \inf_{(\delta, v, \hat{\tau}_n) \in V_h \times U_{h,0} \times U_{h,1}} \left(\|\varepsilon^* - \delta\|_V + \|u^* - v\|_{U_0} + \|\hat{\sigma}_n^* - \hat{\tau}_n\|_{U_1} \right) \end{aligned}$$

DPG eigenvalue approximation (ultra weak)

DPG ultra weak formulation fits within the abstract setting

$$
\begin{aligned}
&U = U_0 \times U_1, \;\; U_0 = L^2(\Omega) \\
&U_1 = L^2(\Omega)^2 \times H_0^{1/2}(\partial\Omega_h) \times H^{-1/2}(\partial\Omega_h) \\
&\mathcal{H} = U_0 \\
&V = H^1(\Omega_h) \times \mathbf{H}(\mathrm{div};\Omega_h) \\
&b(u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n; v, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\Omega_h} - (u, \mathrm{div}\, \boldsymbol{\tau})_{\Omega_h} + \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial\Omega_h} \\
&\qquad\qquad\qquad\qquad - (\boldsymbol{\sigma}, \nabla v)_{\Omega_h} + \langle v, \hat{\sigma}_n \rangle_{\partial\Omega_h} \\
&m(u; v, \boldsymbol{\tau}) = (u, v)_{\Omega_h},
\end{aligned}
$$

DPG eigenvalue approximation (ultra weak)

$k \geq 0$:

$$U_h := P_k(\Omega_h) \times P_k(\Omega_h; \mathbb{R}^2) \times S_0^{k+1}(\partial\Omega_h) \times P_k(\partial\Omega_h)$$
$$V_h := P_{k+2}(\Omega_h) \times P_{k+2}(\Omega_h; \mathbb{R}^2)$$

with

$$S_0^{k+1}(\partial\Omega_h) := \gamma_0(S_0^{k+1}(\Omega_h) \cap H_0^1(\Omega)).$$

A priori error analysis (Führer, 2018)

$$\|u - u_h\|_{U_0} \leq Ch^s \|f\|_{L^2(\Omega)}$$

$\Rightarrow$ uniform convergence

DPG eigenvalue approximation (ultra weak)

- Recall: find $u_h \in U_h$ and $\varepsilon_h \in V_h$ such that

$$\begin{cases} (\varepsilon_h, v_h)_V + b(u_h, v_h) = \ell(v_h) & \forall v_h \in V_h \\ \overline{b(z_h, \varepsilon_h)} = 0 & \forall z_h \in U_h. \end{cases}$$

- C. Carstensen, L. Demkowicz, and J. Gopalakrishnan, A posteriori error control for DPG methods, SIAM J. Numer. Anal. 52 (2014)
- Consider $B, M : U \to V'$ defined as

$$\begin{aligned} (Bu)(v) &:= b(u, v) && \forall u \in U,\ \forall v \in V \\ (Mu)(v) &:= m(u_0, v) && \forall u \in U,\ \forall v \in V, \end{aligned}$$

- *global* indicator

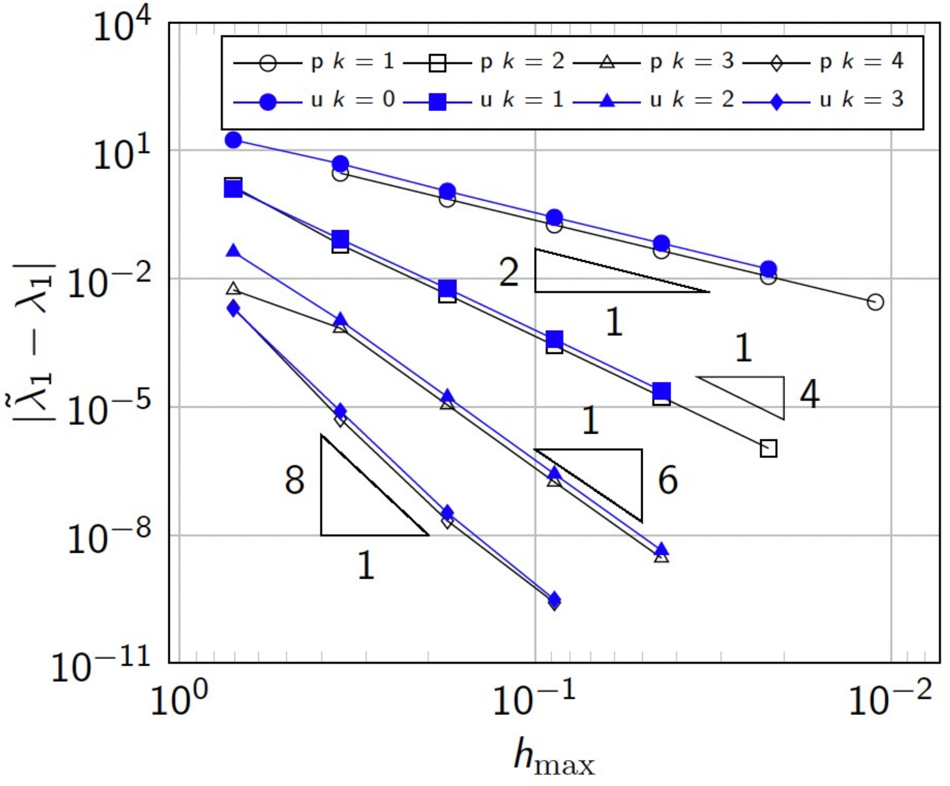
$$\eta = \|\varepsilon_h\| = \|\lambda_h M u_h - B u_h\|_{V'}$$

- Efficiency and reliability up to higher order term $\lambda u_0 - \lambda_h u_{0,h}$.

DPG eigenvalue approximation : numerical results

$\Omega = [0,1]^2$

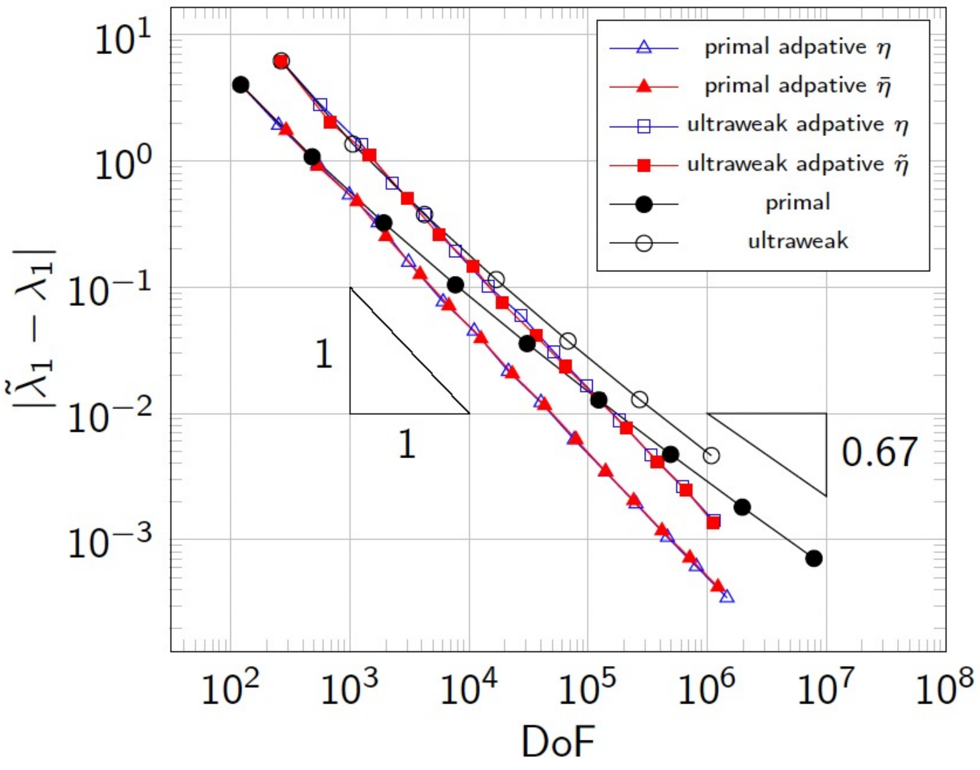
exact solution: $\lambda_1 = 2\pi^2$, p = primal, u = ultra weak.



L-shaped domain

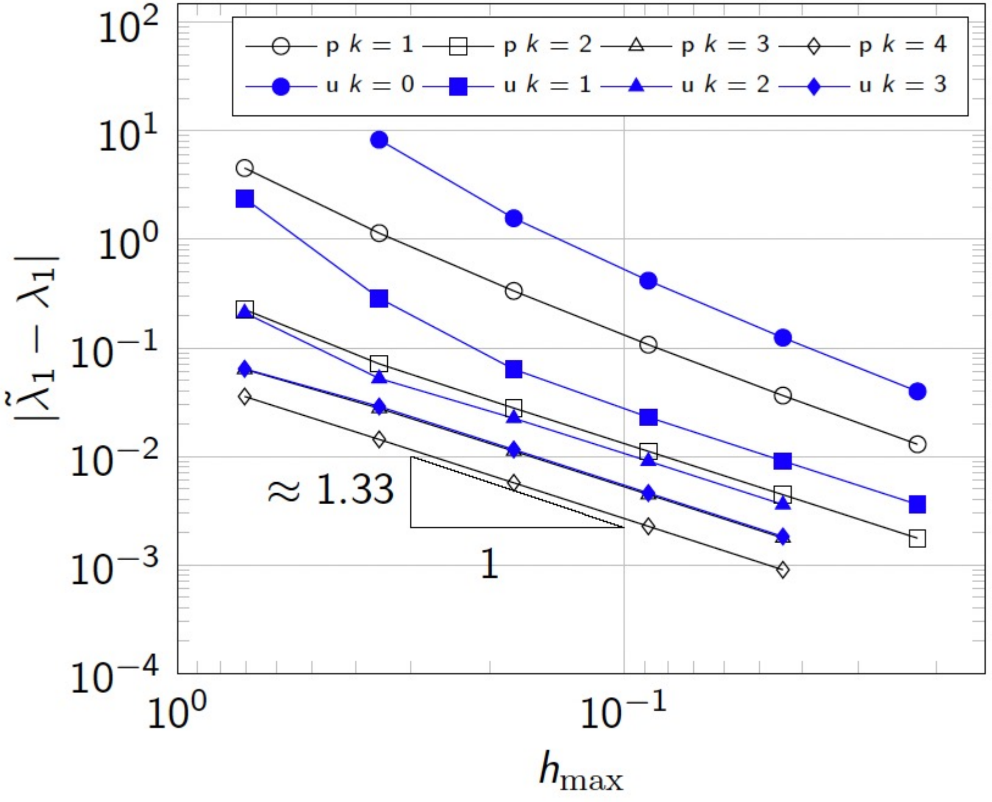
reference values $\lambda_1 = 9.639723844871536$

For bulk parameter $\theta = 0.5$



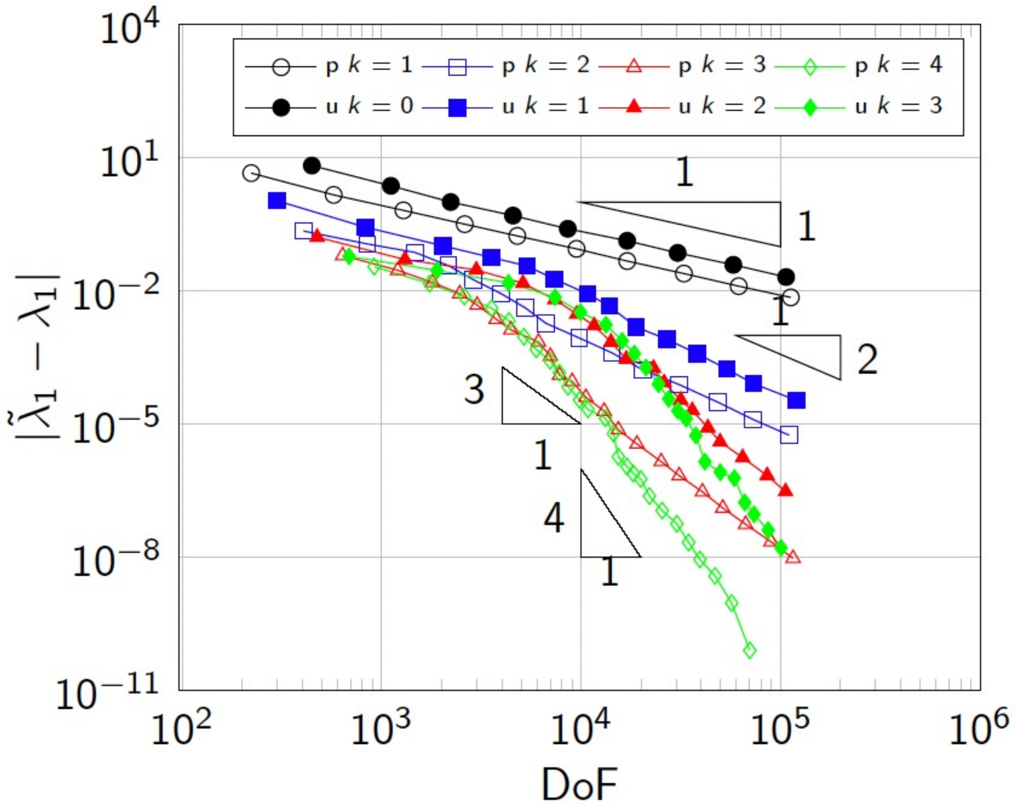
DPG eigenvalue approximation : numerical results

Convergence rates for higher order elements
(p = primal, u= ultraweak)



DPG eigenvalue approximation : numerical results

Adaptive convergence rates for the L-shaped domain for higher order elements (p= primal, u = ultraweak)



DPG eigenvalue approximation

Efficiency-ratio: $\frac{\eta}{\|\tilde{u}-u\|_U}$.

DoF	Efficiency-ratio L-shaped	DoF	Efficiency-ratio Slit
120	7.759	160	8.406
290	7.557	375	8.218
535	7.718	755	8.852
1145	8.914	1365	10.191
2005	9.064	2390	11.435
3875	9.290	4330	12.757
6765	9.714	7490	14.161
12565	9.843	13390	16.884
23075	9.471	23020	17.713
43055	10.200	41345	18.275
79080	9.998	72360	17.590
140000	9.258	124155	14.341

Outline

- Laplace eigenvalue approximation with LSFEM
- Laplace eigenvalue approximation with DPG
- Vibration of elastic structures
- Outlook and conclusions

Vibration of elastic structures

- Polytopal domain $\Omega \in \mathbb{R}^d$ $(d = 2, 3)$, $\partial\Omega = \Gamma_D \cup \Gamma_N$
  find d-by-d stress tensor $\boldsymbol{\sigma}$ and a displacement $\mathbf{u}$ such that

$$\begin{aligned} \mathcal{A}\boldsymbol{\sigma} - \underline{\varepsilon}(\mathbf{u}) &= 0 && \text{in } \Omega \\ \operatorname{div} \boldsymbol{\sigma} &= -\mathbf{f} && \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D \\ \boldsymbol{\sigma}\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N, \end{aligned} \qquad \text{with } \mathcal{A}\boldsymbol{\tau} = \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\tau})\underline{\mathbf{I}}\right)$$

- Cai/Starke 2004

$$\mathcal{F}(\boldsymbol{\tau}, \mathbf{v}; \mathbf{f}) = \|\mathcal{A}\boldsymbol{\sigma} - \underline{\varepsilon}(\mathbf{u})\|_0^2 + \|\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}\|_0^2$$

- Three-field formulation

$$\mathcal{G}(\boldsymbol{\tau}, \mathbf{v}, \varphi; \mathbf{f}) = \|\mathcal{A}\boldsymbol{\tau} - \nabla \mathbf{v} + (-\mathbf{1})^{\mathbf{d}}\chi\varphi\|_0^2 + \|\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}\|_0^2 + \|\operatorname{as} \boldsymbol{\tau}\|_0^2$$

Vibration of elastic structures

- two-field formulation

find $\omega \in \mathbb{C}$ such that for a non vanishing $\mathbf{u} \in H^1_{0,D}(\Omega)^d$ and for some $\boldsymbol{\sigma} \in \underline{\mathbf{X}}_N$

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}, \mathcal{A}\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau}) - (\mathcal{A}\boldsymbol{\tau}, \underline{\varepsilon}(\mathbf{u})) = -\omega(\mathbf{u}, \operatorname{div}\boldsymbol{\tau}) \\ -(\mathcal{A}\boldsymbol{\sigma}, \underline{\varepsilon}(\mathbf{v}) + (\underline{\varepsilon}(\mathbf{u}), \underline{\varepsilon}(\mathbf{v})) = \mathbf{0} \end{cases}$$

for all $\boldsymbol{\tau} \in \underline{\mathbf{X}}_N$ and $\mathbf{v} \in \mathbf{H}^1_{0,D}(\boldsymbol{\Omega})^{\mathbf{d}}$ holds.

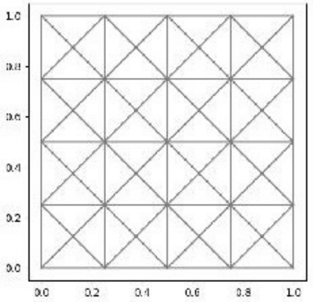
Vibration of elastic structures

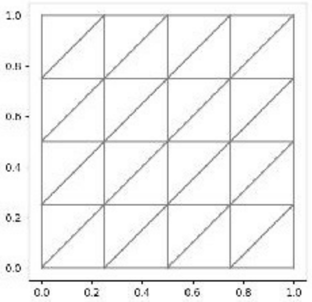
- three-field formulation

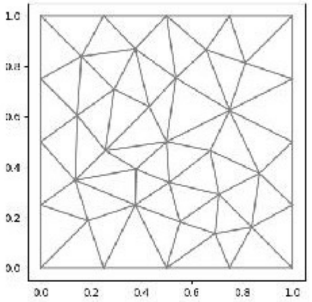
find $\omega \in \mathbb{C}$ such that for a non vanishing $\mathbf{u} \in H^1_{0,D}(\Omega)^d$ and for some $\boldsymbol{\sigma} \in \underline{\mathbf{X}}_N$ and $\psi \in \bar{L}^2(\Omega)$ it holds

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}, \mathcal{A}\boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\sigma}, \operatorname{div} \boldsymbol{\tau}) + (\operatorname{as}(\boldsymbol{\sigma}), \operatorname{as}(\boldsymbol{\tau})) \\ \qquad - (\mathcal{A}\boldsymbol{\tau}, \underline{\varepsilon}(\mathbf{u})) + (-1)^d (\mathcal{A}\boldsymbol{\tau}, \chi\psi) = -\omega(\mathbf{u}, \operatorname{div} \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \underline{\mathbf{X}}_N \\ -(\mathcal{A}\boldsymbol{\sigma}, \underline{\varepsilon}(\mathbf{v})) + (\underline{\varepsilon}(\mathbf{u}), \underline{\varepsilon}(\mathbf{v})) - (-1)^d (\chi\psi, \nabla \mathbf{v}) = \mathbf{0} & \forall \mathbf{v} \in \mathbf{H}^1_{0,D}(\boldsymbol{\Omega})^d \\ (-1)^d (\mathcal{A}\boldsymbol{\sigma}, \chi\varphi) - (-1)^d (\chi\varphi, \nabla \mathbf{u}) + (\chi\psi, \chi\varphi) = 0 & \forall \varphi \in \bar{L}^2(\Omega). \end{cases}$$

- Numerical results on the unit square $\omega = 52.344691168$







Eigenvalues, two-field formulation, $\Sigma_h = RT_1$, $U_h = P_2^c$

	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$
C	52.618734	52.400609 (3.9)	52.362201 (4.0)	52.351749 (4.1)	52.348048 (4.1)
R	54.132943	52.751624 (3.7)	52.480276 (3.8)	52.401472 (3.9)	52.372369 (3.9)
N	52.744298	52.435687 (3.6)	52.368139 (4.7)	52.354017 (4.1)	52.349733 (3.4)

Eigenvalues, three-field formulation, $\Sigma_h = RT_1$, $U_h = P_2^c$, $\Phi_h = P_1^d$

	$N=4$	$N=6$ (rate)	$N=8$ (rate)	$N=10$ (rate)	$N=12$ (rate)
C	52.523637	52.377459 (4.2)	52.353859 (4.4)	52.348025 (4.5)	52.346144 (4.6)
R	53.712947	52.621373 (3.9)	52.426543 (4.2)	52.375437 (4.4)	52.358317 (4.5)
N	52.630390	52.398013 (4.1)	52.355912 (5.4)	52.348310 (5.1)	52.347239 (1.9)

Numerical results on the unit square

First five eigenvalues, three-field scheme, non-structured mesh for 10-th refinement

#	Value
1	52.34830987078562
2	$92.163865261631784 - 0.000422328750065i$
3	$92.163865261631784 + 0.000422328750065i$
4	128.2902654382040
5	154.3710922938166

11-th refinement

$\omega = 52.344691168$

#	Value
1	52.346475661045424
2	92.147313995882541
3	92.151062887227738
4	128.2536615472890
5	154.2967136612170

Vibration of elastic structures

Numerical results on the L-shaped domain

Overkill solution: $\omega = 32.13269464746$.

Eigenvalues, two-field formulation, $\Sigma_h = RT_1$, $U_h = P_2^c$

Mesh	$N = 4$	$N = 8$ (rate)	$N = 16$ (rate)	$N = 32$ (rate)
Uniform	35.606285	31.937374 (4.2)	31.871123 (-0.4)	31.983573 (0.8)

Eigenvalues, three-field formulation, $\Sigma_h = RT_1$, $U_h = P_2^c$, $\Phi_h = P_1^d$

Mesh	$N = 4$	$N = 8$ (rate)	$N = 16$ (rate)	$N = 32$ (rate)
Uniform	34.132843	31.491151 (1.6)	31.677105 (0.5)	31.888816 (0.9)

Vibration of elastic structures

- Numerical results on the L-shaped domain

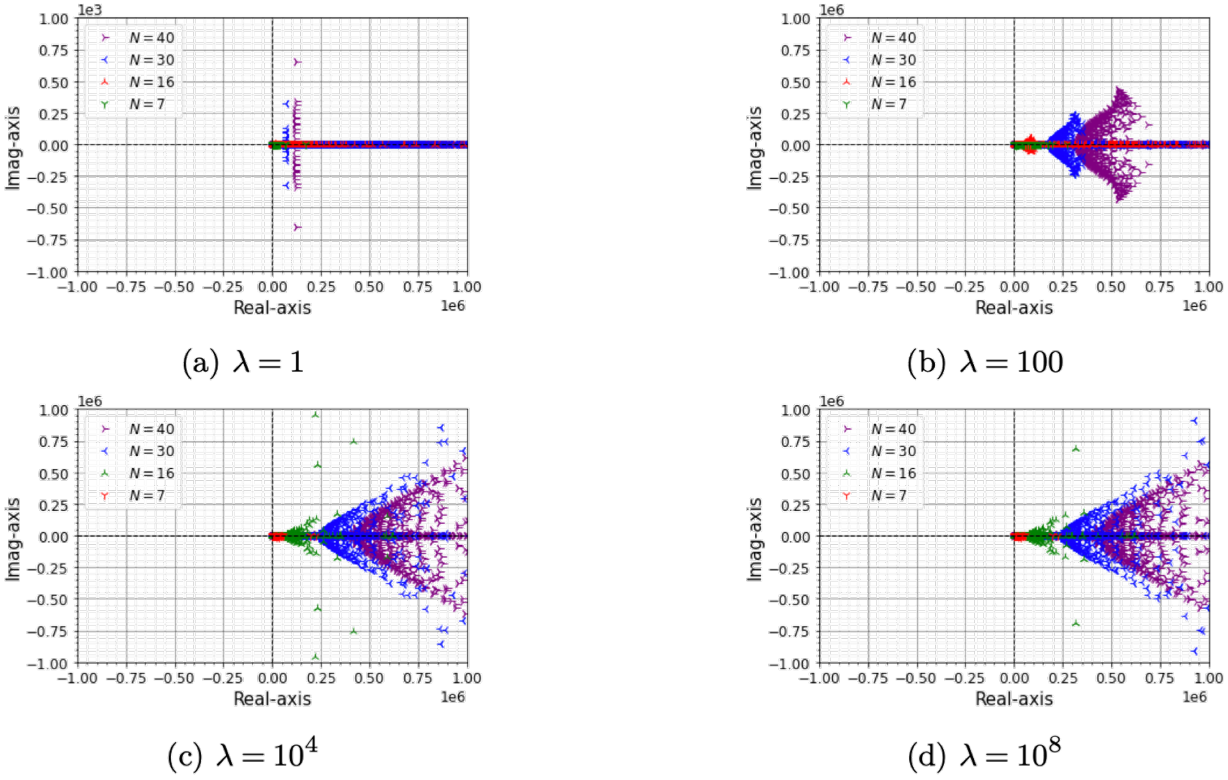
Overkill solution: $\omega = 32.13269464746.$

Eigenvalues, three-field scheme, $N = 8$

#	Value
38	339.9524318713583
39	346.0018703851194
40	$350.8454478342160 - 2.5574107928386i$
41	$350.8454478342160 + 2.5574107928386i$
42	359.0078935078376
43	378.8779264703741

Vibration of elastic structures

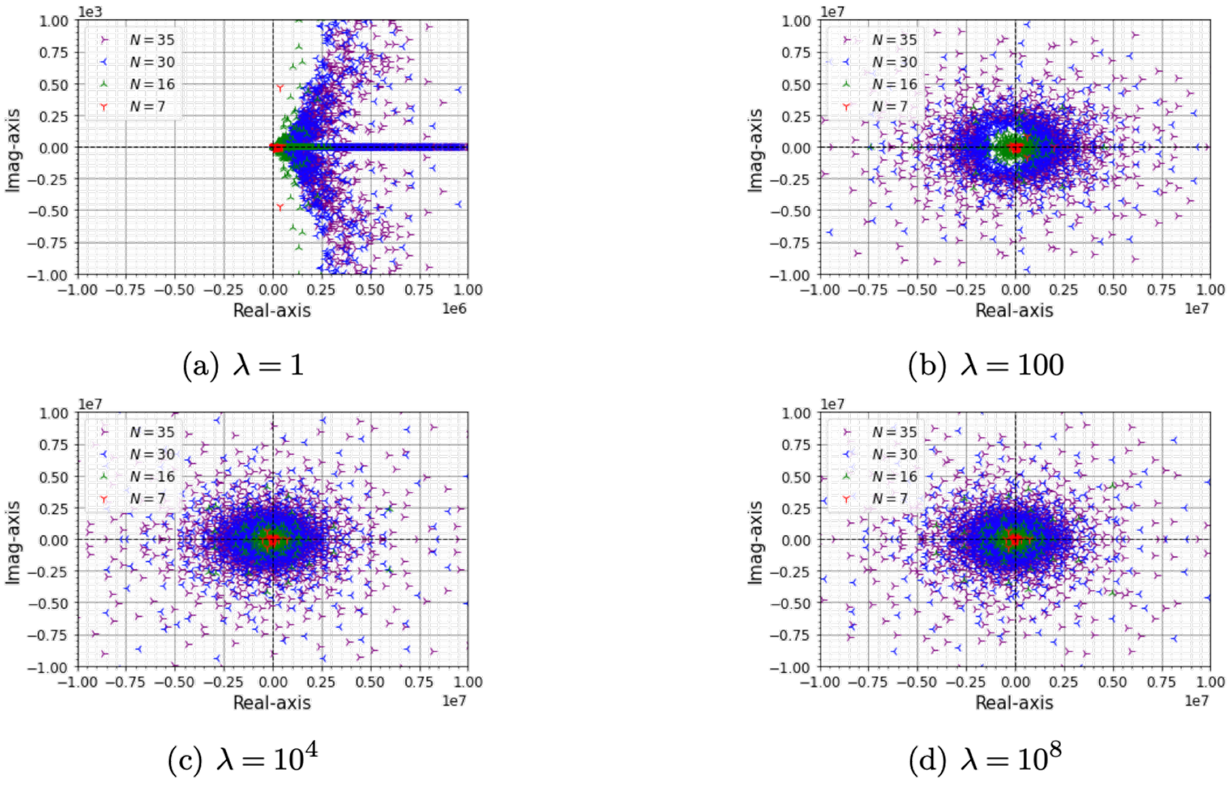
- Crucial role of the mesh structure (two-field, non-robust)



(a) $\lambda = 1$ (b) $\lambda = 100$

(c) $\lambda = 10^4$ (d) $\lambda = 10^8$

Eigenvalues of a square on a refined Crossed mesh



(a) $\lambda = 1$ (b) $\lambda = 100$

(c) $\lambda = 10^4$ (d) $\lambda = 10^8$

Eigenvalues of a square on a refined Nonuniform mesh