USING POLYDPG TO SIMULATE NONLINEAR MECHANICS OF ELASTOMERS

Jaime Mora Paz

Konrad Lorenz Foundation's School of Mathematics and Engineering

Pontificia Universidad Católica de Chile - MINRES/LS 5
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Outline

Practical DPG in L^q $(q \geq 2)$

Nonlinear elastostatics (hyperelasticity) problem

Linearization and variational formulations

Practical application

Conclusions

Extension of practical DPG to $L^q(q \geq 2)$

Let $q \geq 2$ and $q^* = q/(q-1)$. We denote the trial and test spaces by $\mathcal{U}^{(q)}$ and $\mathcal{V}^{(q^*)}$. $b : \mathcal{U}^{(q)} \times \mathcal{V}^{(q^*)}$ is bilinear and $\ell \in \mathcal{V}^{(q^*)\prime}$.

CONTINUOUS PROBLEM: Find $\mathfrak{u} \in \mathcal{U}^{(q)}$ such that for all $\mathfrak{v} \in \mathcal{V}^{(q^*)}$

$$b(\mathfrak{u}, \mathfrak{v}) = \ell(\mathfrak{v}). \qquad \mathcal{B}^{(q,q^*)}\mathfrak{u} = \ell$$

Finite-dimensional subspace $\mathcal{U}^h \subset \mathcal{U}^{(q)}$ of dimension N_h.

DISCRETE PROBLEM: Find $\mathfrak{u}_h \in \mathcal{U}^h$ such that

$$b(\mathfrak{u}_h, \mathfrak{v}_h) = \ell(\mathfrak{v}_h), \qquad (\mathcal{B}^{(q,q^*)}\mathfrak{u}_h)|_{\mathcal{V}^h} = \ell|_{\mathcal{V}^h}$$

for all $\mathfrak{v}_h \in \mathcal{V}^h$, where $\mathcal{V}^h \subset \mathcal{V}^{(q^*)}$ is of dimension N_h and guarantees stability.

Extension of practical DPG to $L^q(q \geq 2)$

Bounded domain $\Rightarrow \mathcal{U}^{(q)} \hookrightarrow \mathcal{U}^{(2)}, \mathcal{V}^{(2)} \xhookrightarrow{\eta^{(q^*)}} \mathcal{V}^{(q^*)}$.
Enriched test space of dim $N_r \geq N_h$ is chosen s.t. $\mathcal{V}^r \xhookrightarrow{\iota} \mathcal{V}^{(2)}$, hence

$$\mathcal{V}^r \xhookrightarrow{\eta^{(q^*)} \circ \iota} \mathcal{V}^{(q^*)} .$$

Let $\mathcal{R}_{\mathcal{V}^r} : \mathcal{V}^r \to \mathcal{V}^{r\,\prime}$ be the Riesz map of $\mathcal{V}^r$ with the $\mathcal{V}^{(2)}$ inner product.
Practical trial-to-test mapping $\mathbf{T}^r : \mathcal{U} \to \mathcal{V}^r$ defined by

$$\mathbf{T}^r := \mathcal{R}_{\mathcal{V}^r}^{-1} \circ \iota^{\mathsf{T}} \circ \eta^{(q^*)\,\mathsf{T}} \circ \mathcal{B}^{(q,q^*)} ,$$

$$\mathfrak{v}_h = \mathbf{T}^r \mathfrak{w}_h \in \mathcal{V}^r \subset \mathcal{V}^{(2)} \quad \iff \quad (\mathfrak{v}_h, \delta\mathfrak{v}^r)_{\mathcal{V}^{(2)}} = b(\mathfrak{w}_h, \delta\mathfrak{v}^r) \quad \forall \delta\mathfrak{v}^r \in \mathcal{V}^r$$

Thus, we define the test space $\mathcal{V}^h$ by

$$\mathcal{V}^h := \mathbf{T}^r(\mathcal{U}^h) \subset \mathcal{V}^r \subset \mathcal{V}^{(2)} \subset \mathcal{V}^{(q^*)}$$

Assumptions to prove stability

- *The linear operator* $\mathcal{B}^{(q,q^*)} : \mathfrak{U}^{(q)} \to \mathcal{V}^{(q^*)\,\prime}$ *associated to the bilinear form* b *is continuous, i.e., there is a constant* $M^{(q^*)} > 0$ *such that*

$$\|\mathcal{B}^{(q,q^*)}\mathfrak{u}\|_{\mathcal{V}^{(q^*)\,\prime}} = \sup_{\mathfrak{v}\in\mathcal{V}^{(q^*)}} \frac{|b(\mathfrak{u},\mathfrak{v})|}{\|\mathfrak{v}\|_{\mathcal{V}^{(q^*)}}} \leq M^{(q^*)}\|\mathfrak{u}\|_{\mathfrak{U}^{(q)}}\,, \qquad \forall \mathfrak{u} \in \mathfrak{U}^{(q)}.$$

- *There exists a constant* $\gamma > 0$ *satisfying*

$$\inf_{\mathfrak{u}\in\mathfrak{U}^{(q)}} \sup_{\mathfrak{v}\in\mathcal{V}^{(q^*)}} \frac{|b(\mathfrak{u},\mathfrak{v})|}{\|\mathfrak{u}\|_{\mathfrak{U}^{(q)}}\|\mathfrak{v}\|_{\mathcal{V}^{(q^*)}}} = \gamma.$$

- *There exists a Fortin operator* $\Pi_F^{(q^*)} : \mathcal{V}^{(q^*)} \to \mathcal{V}^r$ *with the following two properties*

$$\begin{cases} b(\mathfrak{u}_h, \mathfrak{v} - \Pi_F^{(q^*)}\mathfrak{v}) & = 0 \qquad \forall \mathfrak{v} \in \mathcal{V}^{(q^*)}, \mathfrak{u}_h \in \mathfrak{U}^h \quad \textit{(orthogonality)}, \\ \|\Pi_F^{(q^*)}\mathfrak{v}\|_{\mathcal{V}^{(q^*)}} & \leq C_F^{(q^*)}\|\mathfrak{v}\|_{\mathcal{V}^{(q^*)}} \qquad \forall \mathfrak{v} \in \mathcal{V}^{(q^*)} \quad \textit{(continuity)}. \end{cases}$$

In conjunction with some inverse estimates for the enriched test space, and some usual tricks for conforming spaces, we can show stability for the discrete problem.

Finite elastostatics

Elastic body: $\exists$ a response function $\mathcal{K} : \Omega \times \mathbb{M}_+ \to \mathbb{M}$ s.t. $\mathbf{P}(\boldsymbol{X}) = \mathcal{K}(\boldsymbol{X}, \mathbf{F}(\boldsymbol{X}))$

Hyperelasticity: there exists a stored energy density $\widetilde{\mathcal{W}} : \Omega \times \mathbb{M}_+ \to \mathbb{R}$ s.t. $[\mathcal{K}(\boldsymbol{X}, \mathbf{F})]_{iJ} := \frac{\partial \widetilde{\mathcal{W}}}{\partial \mathbf{F}_{iJ}}(\boldsymbol{X}, \mathbf{F})$.

Hyperelasticity + MFI + Isotropy: $\exists \grave{\mathcal{W}} : \Omega \times (0, \infty)^3 \to \mathbb{R}$ satisfying $\widetilde{\mathcal{W}}(\cdot, \mathbf{F}) = \grave{\mathcal{W}}(\cdot, \lambda_1, \lambda_2, \lambda_3)$.

Behavior at large strain $\widetilde{\mathcal{W}}(\cdot, \mathbf{F}) \to +\infty$ as $\det \mathbf{F} \to 0^+$, $\mathbf{F} \in \mathbb{M}_+$.

Coerciveness $\widetilde{\mathcal{W}}(\boldsymbol{X}, \mathbf{F}) \geq \alpha \left(|\mathbf{F}|^p + |\mathrm{Cof}\mathbf{F}|^q + |\det \mathbf{F}|^r \right) + \beta$.

Polyconvexity: $\exists$ a convex function $\mathbb{W} : \Omega \times \mathbb{M} \times \mathbb{M} \times (0, +\infty)$ s.t. $\forall \mathbf{F} \in \mathbb{M}_+$, $\widetilde{\mathcal{W}}(\cdot, \mathbf{F}) = \mathbb{W}(\cdot, \mathbf{F}, \mathrm{Cof}\mathbf{F}, \det \mathbf{F})$.

Elasticity tensor: $\mathsf{A}(\mathbf{F}) = \frac{\partial^2 \widetilde{\mathcal{W}}}{\partial \mathbf{F}\, \partial \mathbf{F}}(\mathbf{F}), \quad \mathsf{A}_{iIjJ} = \mathsf{A}_{jJiI}$.

Strong ellipticity (Legendre-Hadamard): $\widetilde{\mathbf{q}}_i \mathbf{q}_I [\mathsf{A}(\mathbf{F})]_{iIjJ} \widetilde{\mathbf{q}}_j \mathbf{q}_J > 0$

Nonlinear problem

From the first principles and the definitions above we get:

General form

$$\begin{cases} -\operatorname{Div}\mathbf{P} &= \rho_0 \boldsymbol{f}_0 & \text{in} & \Omega, \\ \mathbf{P} - \mathcal{K}(\mathbf{I}+\mathbf{D}) &= \mathbf{0} & \text{in} & \Omega, \\ \mathbf{D} - \operatorname{Grad}\boldsymbol{u} &= \mathbf{0} & \text{in} & \Omega, \\ \\ \boldsymbol{u} &= \boldsymbol{u}_{bc} & \text{on} & \Gamma_u, \\ \mathbf{P}\mathbf{n} &= \boldsymbol{t}_{bc} & \text{on} & \Gamma_t. \end{cases}$$

Reduced form

$$\begin{cases} -\operatorname{Div}\mathcal{K}(\mathbf{I}+\operatorname{Grad}\boldsymbol{u}) &= \rho_0 \boldsymbol{f}_0 & \text{in} & \Omega, \\ \\ \boldsymbol{u} &= \boldsymbol{u}_{bc} & \text{on} & \Gamma_u, \\ \mathcal{K}(\mathbf{I}+\operatorname{Grad}\boldsymbol{u})\mathbf{n} &= \boldsymbol{t}_{bc} & \text{on} & \Gamma_t. \end{cases}$$

Energy minimization principle

$$\mathcal{J}(\boldsymbol{v}) = \int_\Omega \widetilde{\mathcal{W}}(\boldsymbol{X}, \mathbf{I} + \operatorname{Grad}\boldsymbol{v}(\boldsymbol{X}))\, d\boldsymbol{X} - \int_\Omega \rho_0 \boldsymbol{f}_0 \cdot \boldsymbol{v}\, d\boldsymbol{X} - \int_{\Gamma_t} \boldsymbol{t}_{bc} \cdot \boldsymbol{v}\, dA.$$

Theorem (Global existence of minimizers[1])

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let $\widetilde{\mathcal{W}} : \Omega \times \mathbb{M}_+$ have the properties:

- *(a)* **Polyconvexity**
- *(b)* **Behavior as $\det \mathbf{F} \to \mathbf{0}^+$**
- *(c)* **Coerciveness**

Let boundary $\Gamma = \partial\Omega$ be partitioned into Γ_u and Γ_t with $\mathrm{meas}(\Gamma_u) > 0$. Let $\boldsymbol{\varphi}_{bc} : \Gamma_u \to \mathbb{R}^3$ be a measurable function such that the set of **kinematically admissible motions:**

$$\begin{aligned}\mathbf{\Phi} := \{\boldsymbol{\psi} \in W^{1,p}(\Omega;\mathbb{R}^3) \ : \quad & \mathrm{Cof}(\mathrm{Grad}\,\boldsymbol{\psi}) \in L^q(\Omega;\mathbb{M}), \quad \det(\mathrm{Grad}\,\boldsymbol{\psi}) \in L^r(\Omega), \\ & \boldsymbol{\psi} = \boldsymbol{\varphi}_{bc} \text{ a.e. on } \Gamma_u, \quad \det \mathrm{Grad}\,\boldsymbol{\psi} > 0 \text{ a.e. in } \Omega\}\end{aligned}$$

is non-empty. Let θ and σ be selected s.t., for $\boldsymbol{f}_0 \in L^\theta(\Omega;\mathbb{R}^3)$ and $\boldsymbol{t}_{bc} \in L^\sigma(\Gamma_t;\mathbb{R}^3)$,

$$l \ : W^{1,p}(\Omega;\mathbb{R}^3) \ni \boldsymbol{\psi} \to l(\boldsymbol{\psi}) := \int_\Omega \rho_0 \boldsymbol{f}_0 \cdot \boldsymbol{\psi} + \int_{\Gamma_t} \boldsymbol{t}_{bc} \cdot \boldsymbol{\psi}$$

is continuous. Then, there exists $\boldsymbol{\varphi} \in \mathbf{\Phi}$ minimizing the total energy $\mathcal{J}$.

[1] Ciarlet, P. (1993). Mathematical elasticity. Vol. 1. Three-dimensional elasticity. NorthHolland.

Local existence and uniqueness: Inverse Function Theorem

Assumptions

- The material is hyperelastic and the response function $\mathcal{K}$ is smooth in $\boldsymbol{X}$ and $\mathbf{F}$.
- The boundary $\partial\Omega$ is C^1.
- Γ_u is nonempty and equals one or more connected components of $\partial\Omega$.
- $\boldsymbol{\varphi}_0 : \Omega \to \mathbb{R}^3$ is a regular deformation (C^3) and **A** at $\boldsymbol{\varphi}_0$ is strongly elliptic.
- The linearized equations at $\boldsymbol{\varphi}_0$ have a unique solution.

Let $\mathfrak{X} = W^{s,p}(\Omega;\mathbb{R}^3)$ and $\mathfrak{Y} = W^{s-2,p}(\Omega;\mathbb{R}^3) \times W^{s-1/p,p}(\Gamma_u;\mathbb{R}^3) \times W^{s-1-1/p,p}(\Gamma_\sigma;\mathbb{R}^3)$. Let $\mathfrak{C} \subset \mathfrak{X}$ be the set of regular deformations. Define $\mathfrak{F} : \mathfrak{C} \to \mathfrak{Y}$ by

$$\mathfrak{F}(\boldsymbol{\varphi}) = \big(-\operatorname{Div}\mathcal{K}(\operatorname{Grad}\boldsymbol{\varphi}), \boldsymbol{\varphi}|_{\Gamma_u}, (\mathcal{K}(\operatorname{Grad}\boldsymbol{\varphi})\mathbf{n})|_{\Gamma_\sigma} \big).$$

Theorem (Local existence and uniqueness of solutions [2])

Make the assumptions (i)-(v) above and assume that $s > 3/p + 1$ ($1 < p < \infty$). Then there are neighborhoods $\mathfrak{U}$ of $\boldsymbol{\varphi}_0$ in $\mathfrak{X}$ and $\mathfrak{Z}$ of $\mathfrak{F}(\boldsymbol{\varphi}_0)$, such that $\mathfrak{F} : \mathfrak{U} \to \mathfrak{Z}$ is one-to-one and onto.

[2] Marsden, J. E. and Hughes, T. J. (1994). Mathematical foundations of elasticity. Dover.

Linearization

Linearized general form

The group variable is $\mathfrak{u}_0 = (\boldsymbol{u}, \mathbf{P}, \mathbf{D})$ and the linearized system, that solves for $\delta\mathfrak{u}_0 = (\delta\boldsymbol{u}, \delta\mathbf{P}, \delta\mathbf{D})$, reads

$$\begin{cases} -\operatorname{Div}\delta\mathbf{P} &= \rho_0 \boldsymbol{f}_0 + \operatorname{Div}\,\mathbf{P}^{[k]} & \text{in} & \Omega, \\ \delta\mathbf{P} - \underbrace{\mathsf{A}(\mathbf{I}+\mathbf{D}^{[k]})}_{\mathsf{A}^{[k]}} : \delta\mathbf{D} &= -\mathbf{P}^{[k]} + \underbrace{\mathcal{K}(\mathbf{I}+\mathbf{D}^{[k]})}_{\mathcal{K}^{[k]}} & \text{in} & \Omega, \\ \delta\mathbf{D} - \operatorname{Grad}\delta\boldsymbol{u} &= -\mathbf{D}^{[k]} + \operatorname{Grad}\boldsymbol{u}^{[k]} & \text{in} & \Omega, \\ \delta\boldsymbol{u} &= \boldsymbol{u}_{bc} - \boldsymbol{u}^{[k]} & \text{on} & \Gamma_u, \\ \delta\mathbf{P}\mathbf{n} &= \boldsymbol{t}_{bc} - (\mathbf{P}^{[k]})\mathbf{n} & \text{on} & \Gamma_t. \end{cases}$$

Linearized reduced form

This linearized system seeks for the increment $\delta\boldsymbol{u}$ satisfying

$$\begin{cases} -\operatorname{Div}\left(\mathsf{A}(\mathbf{I}+\operatorname{Grad}\boldsymbol{u}^{[k]}) : \operatorname{Grad}\delta\boldsymbol{u}\right) &= \rho_0 \boldsymbol{f}_0 + \operatorname{Div}\mathcal{K}(\mathbf{I}+\operatorname{Grad}\boldsymbol{u}^{[k]}) & \text{in} & \Omega, \\ \delta\boldsymbol{u} &= \boldsymbol{u}_{bc} - \boldsymbol{u}^{[k]} & \text{on} & \Gamma_u, \\ \left(\mathsf{A}(\mathbf{I}+\operatorname{Grad}\boldsymbol{u}^{[k]}) : \operatorname{Grad}\delta\boldsymbol{u}\right)\mathbf{n} &= \boldsymbol{t}_{bc} - \mathcal{K}(\mathbf{I}+\operatorname{Grad}\boldsymbol{u}^{[k]})\mathbf{n} & \text{on} & \Gamma_t. \end{cases}$$

Variational formulation for the linearized BVP in $L^q(q > 3)$

$$\begin{cases} \text{Find } (\delta\mathfrak{u}_0, \delta\hat{\mathfrak{u}}) \in \mathcal{U}_0 \times \hat{\mathcal{U}} \text{ such that} \\ b_0^{[k]}(\delta\mathfrak{u}_0, \mathfrak{v}) + \hat{b}^{[k]}(\delta\hat{\mathfrak{u}}, \mathfrak{v}) = \ell_{\mathcal{T}_h}^{[k]}(\mathfrak{v}) \qquad \forall \mathfrak{v} \in \boldsymbol{\mathcal{V}} \end{cases}$$

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Broken Ultraweak Variational Formulation

$$\begin{cases} (\delta\mathfrak{u}_0 \ , \ \delta\hat{\mathfrak{u}}) = (\delta\boldsymbol{u}, \delta\mathbf{P}, \delta\mathbf{D} \ , \ \delta\hat{\boldsymbol{u}}, \delta\hat{\boldsymbol{t}}) \\ \mathcal{U}_0 \ \times \ \hat{\mathcal{U}} = \mathsf{L}^q(\Omega) \ \times \ W_{\Gamma_u}^{1-1/q,q}(\partial\mathcal{T}_h;\mathbb{R}^3) \times W_{\Gamma_t}^{-1/q,q}(\partial\mathcal{T}_h;\mathbb{R}^3) \end{cases}$$

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$$\begin{cases} (\delta\mathfrak{u}_0 \;,\; \delta\hat{\mathfrak{u}}) &= (\delta\boldsymbol{u}, \delta\mathbf{P}, \delta\mathbf{D} \;,\; \delta\hat{\boldsymbol{u}}, \delta\hat{\boldsymbol{t}}) \\ \mathcal{U}_0 \;\times\; \hat{\mathcal{U}} &= \mathsf{L}^q(\Omega) \;\times\; W_{\Gamma_u}^{1-1/q,q}(\partial\mathcal{T}_h; \mathbb{R}^3) \times W_{\Gamma_t}^{-1/q,q}(\partial\mathcal{T}_h; \mathbb{R}^3) \\ \mathfrak{v} &= (\boldsymbol{v}, \boldsymbol{\tau}, \boldsymbol{\chi}) \\ \boldsymbol{\mathcal{V}} &= W^{1,q^*}(\mathcal{T}_h; \mathbb{R}^3) \times W^{q^*}(\mathrm{Div}, \mathcal{T}_h; \mathbb{R}^3) \times L^{q^*}(\Omega; \mathbb{M}) \\ b_0^{[k]}(\delta\mathfrak{u}_0, \mathfrak{v}) &= (\delta\boldsymbol{u} \;,\; \mathrm{Div}\, \boldsymbol{\tau})_{\mathcal{T}_h} + (\delta\mathbf{P} \;,\; \mathrm{Grad}\, \boldsymbol{v} + \boldsymbol{\chi})_{\mathcal{T}_h} + (\delta\mathbf{D} \;,\; \boldsymbol{\tau} - \mathsf{A}^{[k]} : \boldsymbol{\chi})_{\mathcal{T}_h} \\ \hat{b}^{[k]}(\delta\hat{\mathfrak{u}}, \mathfrak{v}) &= -\langle \delta\hat{\boldsymbol{t}}, \mathrm{tr}_{\mathcal{T}_h}^{\mathrm{Grad}} \boldsymbol{v} \rangle_{\partial\mathcal{T}_h} - \langle \delta\hat{\boldsymbol{u}}, \mathrm{tr}_{\mathcal{T}_h}^{\mathrm{Div}} \boldsymbol{\tau} \rangle_{\partial\mathcal{T}_h} \end{cases}$$

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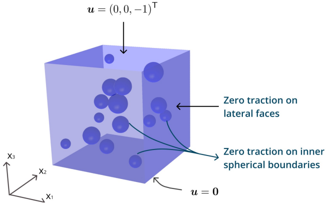
$$\begin{cases} (\delta\mathfrak{u}_0 \ , \ \delta\hat{\mathfrak{u}}) &= (\delta\boldsymbol{u}, \delta\mathbf{P}, \delta\mathbf{D} \ , \ \delta\hat{\boldsymbol{u}}, \delta\hat{\boldsymbol{t}}) \\ \mathcal{U}_0 \ \times \ \hat{\mathcal{U}} &= \mathsf{L}^q(\Omega) \ \times \ W_{\Gamma_u}^{1-1/q,q}(\partial\mathcal{T}_h;\mathbb{R}^3) \times W_{\Gamma_t}^{-1/q,q}(\partial\mathcal{T}_h;\mathbb{R}^3) \\ \mathfrak{v} &= (\boldsymbol{v}, \boldsymbol{\tau}, \boldsymbol{\chi}) \\ \boldsymbol{\mathcal{V}} &= W^{1,q^*}(\mathcal{T}_h;\mathbb{R}^3) \times W^{q^*}(\mathrm{Div}, \mathcal{T}_h;\mathbb{R}^3) \times L^{q^*}(\Omega;\mathbb{M}) \\ b_0^{[k]}(\delta\mathfrak{u}_0, \mathfrak{v}) &= (\delta\boldsymbol{u} \ , \ \mathrm{Div}\,\boldsymbol{\tau})_{\mathcal{T}_h} + (\delta\mathbf{P} \ , \ \mathrm{Grad}\,\boldsymbol{v} + \boldsymbol{\chi})_{\mathcal{T}_h} + (\delta\mathbf{D} \ , \ \boldsymbol{\tau} - \mathsf{A}^{[k]} : \boldsymbol{\chi})_{\mathcal{T}_h} \\ \hat{b}^{[k]}(\delta\hat{\mathfrak{u}}, \mathfrak{v}) &= -\langle \delta\hat{\boldsymbol{t}}, \mathrm{tr}_{\mathcal{T}_h}^{\mathrm{Grad}} \boldsymbol{v} \rangle_{\partial\mathcal{T}_h} - \langle \delta\hat{\boldsymbol{u}}, \mathrm{tr}_{\mathcal{T}_h}^{\mathrm{Div}} \boldsymbol{\tau} \rangle_{\partial\mathcal{T}_h} \\ \ell_{\mathcal{T}_h}^{[k]}(\mathfrak{v}) &= (\rho_0 \boldsymbol{f}_0 \ , \ \boldsymbol{v})_{\mathcal{T}_h} - (\mathbf{D}^{[k]} \ , \ \boldsymbol{\tau})_{\mathcal{T}_h} + \langle \hat{\boldsymbol{t}}^{[k]} \ , \ \mathrm{tr}_{\mathcal{T}_h}^{\mathrm{Grad}} \boldsymbol{v} \rangle_{\Gamma_t} + (-\mathbf{P}^{[k]} + \mathcal{K}^{[k]} \ , \ \boldsymbol{\chi})_{\mathcal{T}_h} \\ & \quad -(\mathbf{P}^{[k]} \ , \ \mathrm{Grad}\,\boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{u}^{[k]} \ , \ \mathrm{Div}\,\boldsymbol{\tau})_{\mathcal{T}_h} + \langle \hat{\boldsymbol{u}}^{[k]} \ , \ \mathrm{tr}_{\mathcal{T}_h}^{\mathrm{Div}} \boldsymbol{\tau} \rangle_{\Gamma_u} \end{cases}$$

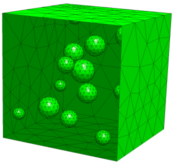
Well-posedness. Assume smooth boundary, $\Gamma_u = \partial\Omega$ and that A is strongly elliptic with C^0 entries. Check conditions on $\hat{b}^{[k]}(\delta\hat{\mathfrak{u}}, \mathfrak{v})$ for broken test spaces.

PRIMAL $\Rightarrow$ STRONG $\Rightarrow$ ULTRAWEAK $\Rightarrow$ BROKEN PRIMAL, BROKEN UW

Problem setup

Length unit: mm. Stress unit: MPa. Domain is the unit cube with 15 spherical voids.





$$\grave{\mathcal{W}}(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{M} \frac{a_i}{\gamma_i}(\lambda_1^{\gamma_i} + \lambda_2^{\gamma_i} + \lambda_3^{\gamma_i}) + \sum_{j=1}^{N} \frac{b_j}{\delta_j}[(\lambda_2\lambda_3)^{\delta_j} + (\lambda_3\lambda_1)^{\delta_j} + (\lambda_1\lambda_2)^{\delta_j}] + G^{vol}(\lambda_1\lambda_2$$

The volumetric term, with $K = 10^3$, is

$$G^{vol}(J) = \frac{1}{2}K\left[\frac{1}{2}(J^2 - 1) - \log J\right].$$

The Ogden model parameters are[3]:

$$M = 2,\ a_1 = 1{,}3,\ \gamma_1 = 0{,}006,\ a_2 = 5,\ \gamma_2 = 1{,}2,$$
$$N = 1,\ b_1 = 2,\ \delta_1 = 0{,}01.$$

[3] Marckmann, G. and Verron, E. (2006). Rubber chemistry and technology, 79(5):835–858.

Implementation details

Let $\mathbf{F}$ have the following singular value decomposition (SVD)

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We use $\boldsymbol{\Lambda}$ to eval energy $\grave{\mathcal{W}}$, stress and elasticity tensor

$$\mathcal{K}(\mathbf{F}) = \widetilde{\mathsf{Q}}\mathcal{K}(\boldsymbol{\Lambda})\mathsf{Q}^{\mathsf{T}}. \qquad \mathsf{k}_{\alpha} = \frac{\partial \grave{\mathcal{W}}}{\partial \lambda_{\alpha}}(\lambda_1, \lambda_2, \lambda_3),$$

for $\alpha = 1, 2, 3$.

$$\mathsf{A}_{iIjJ}(\mathbf{F}) = \mathbf{F}_{iK}\mathbf{F}_{jL}\mathsf{C}_{IJKL}(\mathbf{F}^{\mathsf{T}}\mathbf{F}) + [\mathcal{S}(\mathbf{F}^{\mathsf{T}}\mathbf{F})]_{IJ}\boldsymbol{\delta}_{ij}.$$

$$\begin{aligned}\mathsf{C}_{IJKL} = &\sum_{\alpha}\sum_{\beta} \frac{\partial \mathfrak{s}_{\alpha}}{\partial \mathfrak{e}_{\beta}} (\mathbf{q}_{\alpha})_I(\mathbf{q}_{\alpha})_J(\mathbf{q}_{\beta})_K(\mathbf{q}_{\beta})_L \\ &+ \frac{1}{2}\sum_{\alpha\neq\beta} \left(\frac{\mathfrak{s}_{\beta} - \mathfrak{s}_{\alpha}}{\mathfrak{e}_{\beta} - \mathfrak{e}_{\alpha}}\right) \left[(\mathbf{q}_{\alpha})_I(\mathbf{q}_{\beta})_J(\mathbf{q}_{\alpha})_K(\mathbf{q}_{\beta})_L + (\mathbf{q}_{\alpha})_I(\mathbf{q}_{\beta})_J(\mathbf{q}_{\beta})_K(\mathbf{q}_{\alpha})_L\right],\end{aligned}$$

Implementation details

Homogeneous numerical integration for polyhedra: Let K be a polyhedron with faces $\{F_j\}$, $g : \mathbb{R}^N \to \mathbb{R}$ be homogeneous of degree k.

$$\int_K g(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{F_j \subset \partial K} \frac{b_j}{N+k} \int_{F_j} g(\boldsymbol{x})\, dA \qquad b_j = \mathbf{n}_j \cdot \boldsymbol{x}_{j,0}.$$

Consider term $(\delta\mathbf{D}\,,\, \boldsymbol{\tau} - \mathbf{A}^{[k]} : \boldsymbol{\chi})_{\mathcal{T}_h}$. Although $\mathfrak{u}^{[k]}$ and $\mathfrak{v}$ are polynomials, $\mathbf{A}^{[k]}$ is not. This works perfectly for lowest order ($\{\lambda\}$ are constant, therefore $\mathbf{A}^{[k]}$ is constant.).

Nonlinear Solver: Modif. Newton with load stepping. We assemble with OpenMP, factorize and solve with MUMPS. Line search and BFGS updates are optional.

Line search

$$\mathfrak{u}^{[k+1]} = s^{[k]}\delta\mathfrak{u} + \mathfrak{u}^{[k]}.$$

Step length $s^{[k]}$ is determined by finding a zero of the scalar function

$$\mathcal{G}^{[k]}(s) = \delta\mathsf{d} \cdot \mathfrak{l}(s, \delta\mathfrak{u}).$$

BFGS (Broyden, Fletcher, Goldfarb and Shanno) updates

$$(\widetilde{\mathsf{B}^{[k+1]}_{\text{n-opt}}})^{-1} := (\mathsf{I} + \mathsf{w}\mathsf{v}^{\mathsf{T}})^{\mathsf{T}}(\mathsf{B}^{[k]}_{\text{n-opt}})^{-1}(\mathsf{I} + \mathsf{w}\mathsf{v}^{\mathsf{T}}).$$

Results

Methods to solve linearized problem:

- **Bubnov–Galerkin:** $\delta \boldsymbol{u}_h, \boldsymbol{v}_h \in (\mathcal{P}_c^p(\mathcal{T}_h))^3$.
- **Ultraweak (UW) DPG, p-enrichment with fixed $dp = 2$**.
- **PolyDPG with continuous traces, variable p-enrichment**.

$$\underbrace{\mathfrak{U}^h}_{\delta\mathfrak{u}_h} = \underbrace{(\mathcal{P}_d^{p-1}(\mathcal{T}_h))^3}_{\delta\boldsymbol{u}_h} \times \underbrace{(\mathcal{P}_d^{p-1}(\mathcal{T}_h))^{3\times 3}}_{\delta\mathbf{P}_h} \times \underbrace{(\mathcal{P}_d^{p-1}(\mathcal{T}_h))^{3\times 3}}_{\delta\mathbf{D}_h} \times \underbrace{(\mathcal{P}_c^{p}(\partial\mathcal{T}_h))^3}_{\delta\hat{\boldsymbol{u}}_h} \times \underbrace{(\mathcal{P}_d^{p-1}(\partial\mathcal{T}_h))^3}_{\delta\hat{\boldsymbol{t}}_h}.$$

$$\underbrace{\mathfrak{V}^r}_{\mathfrak{v}^r} = \underbrace{(\mathcal{P}_d^{p+dp}(\mathcal{T}_h))^3}_{\boldsymbol{v}^r} \times \underbrace{(\mathcal{RT}_d^{p+dp}(\mathcal{T}_h))^3}_{\boldsymbol{\tau}^r} \times \underbrace{L^2(\Omega;\mathbb{M})}_{\boldsymbol{\chi}}$$

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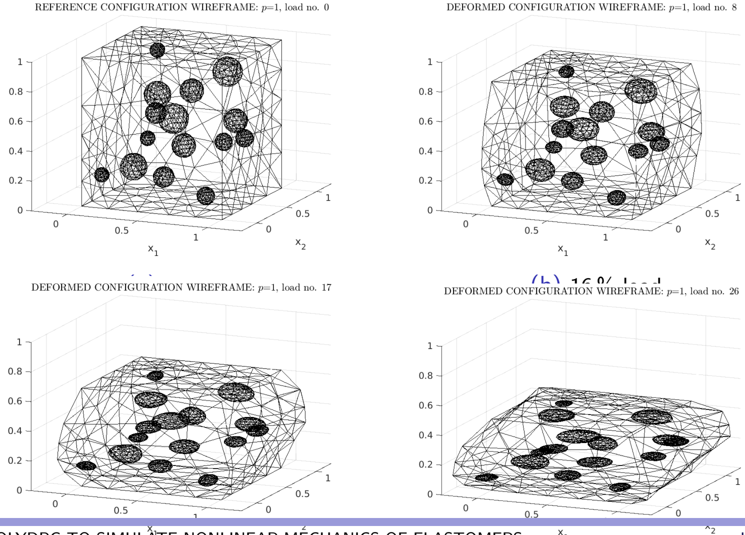
Mesh	Method	p	DOF	System size	Line search	Converged steps	Load step size	Final load
7,453	BG	1	6,036	5,784	Yes	10	0.002	2 %
7,453	UW DPG	1	353,457	46,359	No	6	0.004	2.4 %
7,453 (aggl.)	PolyDPG	1	106,854	21,369	No	26	0.02	52 %

Results with agglomerated elements



(b) 16 % load

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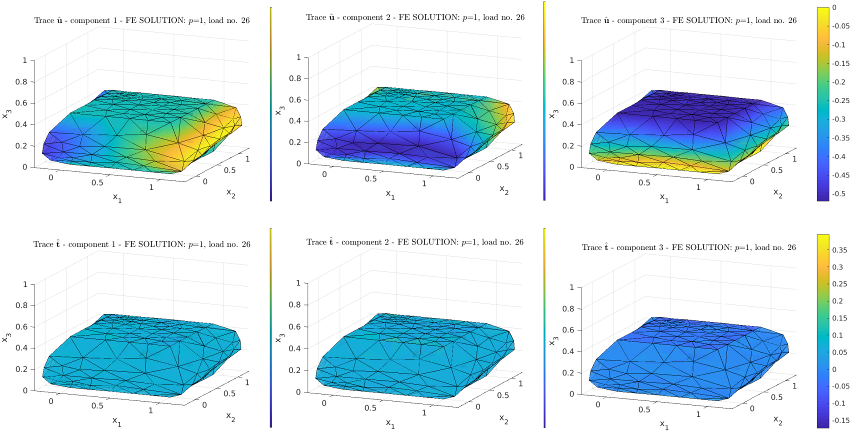

Solution plots at the maximum deformation: (top) displacement trace $\hat{\boldsymbol{u}}_h$; (bottom) traction $\hat{\boldsymbol{t}}_h$.

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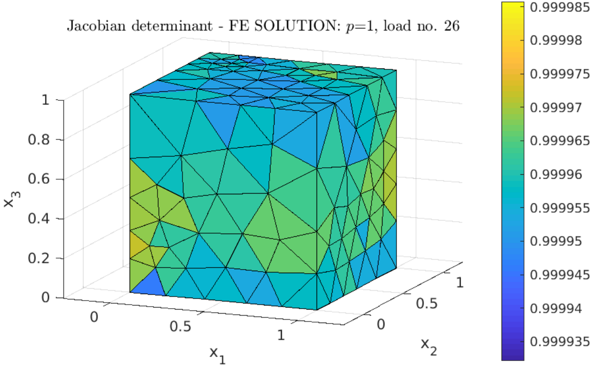

Solution plot of the Jacobian at the maximum deformation (52 % load).

Conclusions

- We have introduced new formulations and DPG discretizations for the linearized equations of elastostatics.
- We have studied a practical application of the new discretization.
- We have shown that the new formulation combined with polyhedral elements can reach larger deformations than other methods.

Ongoing and future work

- Implement other options for the nonlinear solver, especially one that can exploit the MPI implementation of `hp3d` to enable the solution of large nonlinear problems.
- Extend the present work to the simulation of a elastomeric syntactic foam, by adding glass microballoons to the problem setup.

Related Publications

- Mora, J. (2020). *PolyDPG: A discontinuous Petrov-Galerkin methodology for polytopal meshes with applications to elasticity*. Doctoral dissertation, University of Texas at Austin.
- Bacuta, C., Demkowicz, L. Mora, J. & Xenophontos, Ch. (2020). *Analysis of non-conforming DPG methods on polyhedral meshes using fractional Sobolev norms*. Comput. Math. Appl. In press.
- Vaziri Astaneh, A., Fuentes, F., Mora, J., & Demkowicz, L. (2018). *High-order polygonal discontinuous Petrov Galerkin (PolyDPG) methods using ultraweak formulations*. Computer Methods in Applied Mechanics and Engineering, 332, 686-711.
- Mora, J., & Demkowicz, L. (2022). *Discontinuous Petrov-Galerkin Finite Element Method for Hyperelasticity*. In preparation.

THANK YOU FOR YOUR ATTENTION!