A continuous hp-mesh model for DPG finite element schemes with optimal test functions

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MINRES/LS-5

October 5, 2022

Outline

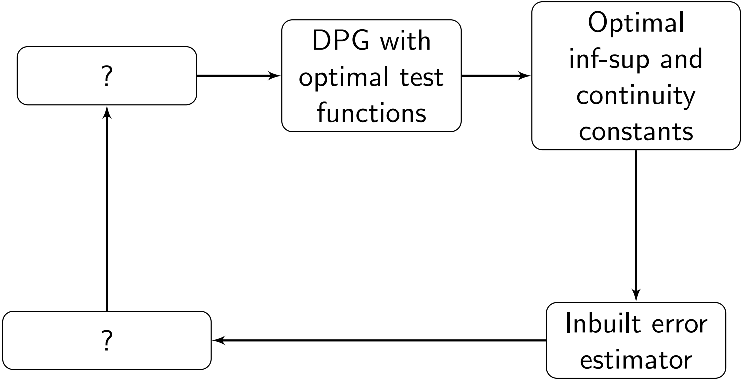
- Background
- Motivation
- Mesh-Metric Duality
- Error Model
- Numerical Results

Background

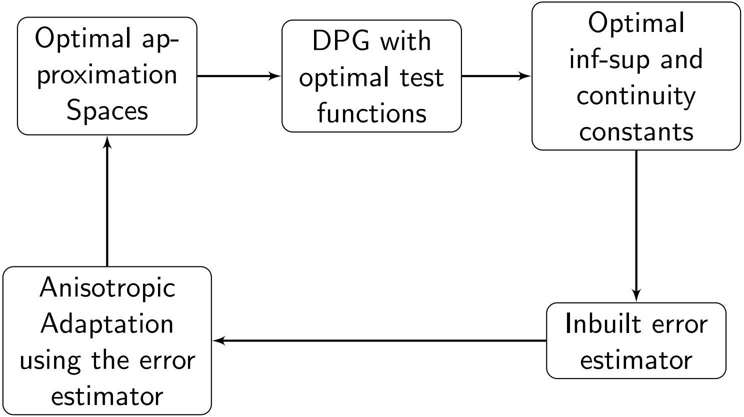
- Discontinuous Petrov-Galerkin schemes with optimal functions.
  - Inbuilt error estimator
  - Polynomial representation of inbuilt error estimator.
- Adaptive Schemes
  - Reduce discretization error
  - Accurate solution on coarse meshes
  - Faster convergence to asymptotic rates

[1] Demkowicz and Gopalakrishnan, 2011, Numerical Methods for Partial Differential Equations
[2] Demkowicz and Gopalakrishnan, 2010, Computer Methods in Applied Mechanics and Engineering
[3] Demkowicz and Heuer, 2013, SIAM Journal on Numerical Analysis

Motivation


?
DPG with optimal test functions
Optimal inf-sup and continuity constants
Inbuilt error estimator
?

Motivation

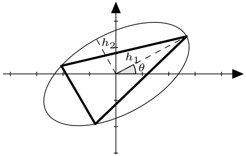

Optimal approximation Spaces
DPG with optimal test functions
Optimal inf-sup and continuity constants
Inbuilt error estimator
Anisotropic Adaptation using the error estimator

Mesh-Metric Duality

Metric-Ellipse correspondence:

$$\sum_{\mathcal{M}} := \{\boldsymbol{x} \in R^2 : \boldsymbol{x}^T \mathcal{M} \boldsymbol{x} \leq 1\}$$

Ellipse-Triangle correspondence



A mesh element is unit with respect to a metric field if each of its edge satisfies the following condition

$$\|e_i\|^2_{\mathcal{M}} = e_i^T \mathcal{M} e_i = C, \quad i = 1, 2, 3$$

Mesh-Metric Duality

$$\mathcal{M} = \begin{bmatrix} \cos(\theta_{\mathcal{M}}) & -\sin(\theta_{\mathcal{M}}) \\ \sin(\theta_{\mathcal{M}}) & \cos(\theta_{\mathcal{M}}) \end{bmatrix}^T \begin{bmatrix} \alpha_{\mathcal{M},1} & 0 \\ 0 & \alpha_{\mathcal{M},2} \end{bmatrix} \begin{bmatrix} \cos(\theta_{\mathcal{M}}) & -\sin(\theta_{\mathcal{M}}) \\ \sin(\theta_{\mathcal{M}}) & \cos(\theta_{\mathcal{M}}) \end{bmatrix}$$

$$\alpha_{\mathcal{M},1} = \frac{1}{h_1^2}, \quad \alpha_{\mathcal{M},2} = \frac{1}{h_2^2}$$

- For $C = 3$, the area of the triangle is related to ellipse by,

$$|\kappa_{\mathcal{M}}| = \frac{3\sqrt{3}}{4} h_1 h_2 = \frac{3\sqrt{3}}{4 d_{\mathcal{M}}}$$

- We define aspect ratio of the element as

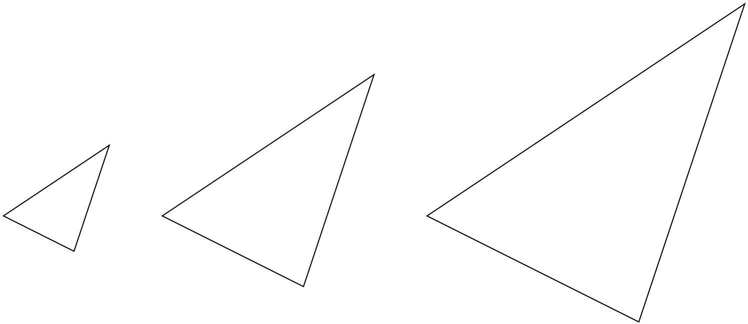
$$\beta_{\mathcal{M}} = \frac{h_1}{h_2}$$

- The triangle is described by $d_{\mathcal{M}}, \beta_{\mathcal{M}}, \theta_{\mathcal{M}}$

Optimization Goal

- Optimize $\beta_{\mathcal{M}}$ and $\theta_{\mathcal{M}}$ by locally minimizing an error estimate.

- We can still have set of similar triangles with same anisotropy.



- $d_{\mathcal{M}}$ is computed in a second optimization step.

Higher Order Error Model

DPG methods with optimal test functions have an inbuilt error estimator:

$$\|U - U_h\|^2_{E,\Omega_h} = \|\mathcal{L}U_h - l\|^2_{\mathbb{V}',\Omega_h} = \|(v,\tau)\|^2_{\mathbb{V},\Omega_h} = \sum_{\kappa\in\mathcal{T}_h} \|(v,\tau)\|^2_{\mathbb{V},\kappa}$$

For scalar convection-diffusion problem with ultra-weak formulation [4], $V \in H^1(\Omega_h)$ and $\boldsymbol{\tau} \in H(div;\Omega_h)$.

$$\|(v,\tau)\|^2_{\mathbb{V},\kappa} = \int_\kappa \underbrace{(v(\boldsymbol{x}))^2 + \boldsymbol{\tau}(\boldsymbol{x})\cdot\boldsymbol{\tau}(\boldsymbol{x}) + |k|(\nabla v(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x}) + (\nabla\cdot\boldsymbol{\tau}(\boldsymbol{x}))^2)}_{e_\kappa(\boldsymbol{x})}\, d\boldsymbol{x}$$

Let the order of approximation for field variables be P and $P+\delta P$ for test space. Then $e_\kappa(\boldsymbol{x})$ is a polynomial of order $2(P+\delta P)$.

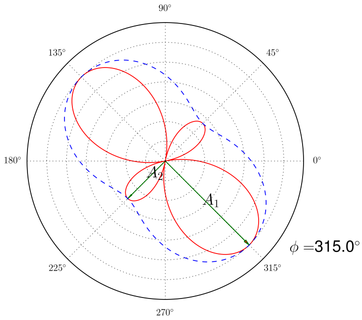
$$e_\kappa(\boldsymbol{x}) = \sum_{i=0}^{2(P+\delta P)} \underbrace{P_{i,\bar{\boldsymbol{x}}}(\boldsymbol{x})}_{\sum_{l=0}^{i} c_l (x-\bar{x})^l (y-\bar{y})^{i-l}}$$

[4] Demkowicz et al., 2012, Applied Numerical Mathematics

Higher Order Error Model

For a polynomial $P_{i,\bar{\boldsymbol{x}}}(\boldsymbol{x}) = \sum_{l=0}^{i} c_l (x-\bar{x})^l (y-\bar{y})^{i-l}$, we have [5]:

$$|P_{i,\bar{\boldsymbol{x}}}(\boldsymbol{x})| \lesssim A_1 \left((\boldsymbol{x}-\bar{\boldsymbol{x}})^T Q_{\phi_i} D_{\rho_i} {Q_{\phi_i}}^T (\boldsymbol{x}-\bar{\boldsymbol{x}}) \right)^{\frac{i}{2}}$$



[5] Dolejsi, 2014, Applied Numerical Mathematics

Higher Order Error Model

$$\int_{\kappa} e_{\kappa}(\boldsymbol{x})\, d\boldsymbol{x} \leq \int_{E_{\kappa}} e_{\kappa}(\boldsymbol{x})\, d\boldsymbol{x} \lesssim \sum_{i=0, i\in z_{ev}^{+}}^{2(P+\delta P)} \int_{E_{\kappa}} A_{i,1} \left((\boldsymbol{x}-\bar{\boldsymbol{x}})^T Q_{\phi_i} D_{\rho_i} {Q_{\phi_i}}^T (\boldsymbol{x}-\bar{\boldsymbol{x}}) \right)^{\frac{i}{2}}\, d\boldsymbol{x}$$

$$\boldsymbol{x}-\bar{\boldsymbol{x}} = Q_{\theta_{\mathcal{M}}} S_{\beta_{\mathcal{M}}} \hat{x} = r Q_{\theta_{\mathcal{M}}} S_{\beta_{\mathcal{M}}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Using polar coordinates, we have

$$\int_{\kappa} e_{\kappa}(\boldsymbol{x})\, d\boldsymbol{x} \leq \int_{E_{\kappa}} e_{\kappa}(\boldsymbol{x})\, d\boldsymbol{x} \lesssim \sum_{i=0, i\in z_{ev}^{+}}^{2(P+\delta P)} \frac{A_{i,1} {\lambda_e}^{\frac{i+2}{2}}}{i+2} \int_0^{2\pi} \left(g_i(\theta; \beta_{\mathcal{M}}, \phi_{\mathcal{M}} - \phi_i) \right)^{\frac{i}{2}}\, d\theta$$

The anisotropic properties $\{\beta_{\mathcal{M}}, \phi_{\mathcal{M}}\}$ are set by locally minimizing the error bound [6].

[6] Dolejsi et al., 2019, SIAM Journal on Scientific Computing

Analytic Optimization (H adaptation)

Assumption [7, 8]:

$$\eta_k^2 \approx \bar{A}_k |k|^{s+1}$$

Error density function [7] :

$$e_d = \bar{A}_k \left(\frac{3\sqrt{3}}{4d_k} \right)^s \qquad e_d(\boldsymbol{x}) = \bar{A}(\boldsymbol{x}) \left(\frac{3\sqrt{3}}{4d(\boldsymbol{x})} \right)^s$$

At asymptotic limit

$$\eta_\Omega^2 = \sum_{k \in \mathcal{T}_h} e_d(\boldsymbol{x}_k)|k| \to \int_\Omega e_d(\boldsymbol{x})\, d\boldsymbol{x}$$

For Goal oriented adaptation $\eta_k = \bar{\eta}_k \cdot \eta_k^\star$ ([9, 10])and $\eta_\Omega \approx O(h^{p+1})$. Also $\bar{A}(\boldsymbol{x}_k) = \frac{\eta_k^2}{|k|^{p+2}}$.

[7] Dolejší et al., 2017, Computers and Mathematics with Applications [8] Venditti and Darmofal, 2003, Journal of Computational Physics [9] Demkowicz and Gopalakrishnan, 2011, SIAM Journal on Numerical Analysis [10] Keith et al., 2019, SIAM Journal on Numerical Analysis

Analytic Optimization (H adaptation)

Problem

Let N be the desired complexity and $e_d(\boldsymbol{x})$ be the error density. We seek a mesh density distribution $d(\boldsymbol{x}) : \Omega \to \mathbb{R}^+$ such that:
(a) $N = \int_\Omega d(\boldsymbol{x})\, d\boldsymbol{x}$.
(b) $E = \int_\Omega e_d(\boldsymbol{x})\, d\boldsymbol{x}$ is minimized.

$$\int_\Omega \delta d(\boldsymbol{x})\, d\boldsymbol{x} = 0$$

$$\delta E = -(p+1)\alpha^{(p+1)} \int_\Omega \bar{A}(\boldsymbol{x}) d(\boldsymbol{x})^{-(p+2)} \delta d\, d\boldsymbol{x} \qquad where \qquad \alpha = \frac{3\sqrt{3}}{4}$$

$$\bar{A}(\boldsymbol{x}) d(\boldsymbol{x})^{-(p+2)} = const.$$

Hence again using the constraint, we obtain

$$d^\star(\boldsymbol{x}) = K\bar{A}(\boldsymbol{x})^{\frac{1}{(p+2)}}, \qquad K = \frac{N}{\int_\Omega \bar{A}(\boldsymbol{x})^{\frac{1}{(p+2)}}\, d\boldsymbol{x}}$$

On substituting the expression for K, we obtain the expression for the optimal density.

$$d^\star(\boldsymbol{x}) = \frac{N\bar{A}(\boldsymbol{x})^{\frac{1}{(p+2)}}}{\int_\Omega \bar{A}(\boldsymbol{x})^{\frac{1}{(p+2)}}\, d\boldsymbol{x}}$$

$$\bar{A}(\boldsymbol{x}_k) = \frac{\eta_k^2}{|k|^{p+2}} \qquad\qquad d^\star(\boldsymbol{x}_k) = \frac{N\eta_k^{\frac{2}{(p+2)}}}{|k|\left(\sum_{k\in\mathcal{T}_h}\eta_k^{\frac{2}{(p+2)}}\right)}$$

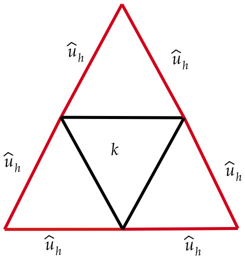
Analytic Optimization (HP adaptation)

- Two step process:
    - Polynomial order selection.
    - Density computation.

Analytic Optimization (HP adaptation)

- Polynomial selection

$$m_{p_k+i} = \left(\frac{E_{p_k+i}}{E_{p_k}}\right)^{\frac{2}{p_k+i+1}} N_{p_k+i}$$



Analytic Optimization (HP adaptation)

Problem

Let N be the desired complexity in terms of degrees of freedom and $e(\boldsymbol{x}, p(\boldsymbol{x}))$ be the error density. We seek a mesh density distribution $d(\boldsymbol{x}) : \Omega \rightarrow \mathbb{R}^+$ for a given polynomial $p(\boldsymbol{x}) : \Omega \rightarrow Z^+$ such that:
(a) $N = \int_\Omega w(\boldsymbol{x}) d(\boldsymbol{x})\, d\boldsymbol{x}$.
(b) $E = \int_\Omega e(d(\boldsymbol{x}), p(\boldsymbol{x}))\, d\boldsymbol{x}$ is minimized.

- Density optimization.
- Local DOF modification.
- $w(\boldsymbol{x}) = \frac{2(p(\boldsymbol{x})+1)(p(\boldsymbol{x})+2)}{3\sqrt{3}}$

Analytic Optimization (HP adaptation)

Density Optimization:

$$\delta N = \int_\Omega w(\boldsymbol{x})\delta d(\boldsymbol{x})d\boldsymbol{x} = 0$$

$$\delta E = \int_\Omega -(p(\mathbf{x})+1)\alpha^{(p(\mathbf{x})+1)}\overline{A}(\mathbf{x})d^{-(p(\mathbf{x})+2)}\delta d\, d\mathbf{x}, \tag{1}$$

$$\frac{(p(\boldsymbol{x})+1)\bar{A}(\boldsymbol{x})}{w(\boldsymbol{x})}d(\boldsymbol{x})^{-(p(\boldsymbol{x})+2)} = K = const.$$

$$d^\star(\boldsymbol{x}) = \left(\frac{(p(\boldsymbol{x})+1)\bar{A}(\boldsymbol{x})}{w(\boldsymbol{x})}\right)^{\frac{1}{(p(\boldsymbol{x})+2)}} K^{-\frac{1}{(p(\boldsymbol{x})+2)}}$$

Scalar Boundary Layer

- Viscous linear equation with source

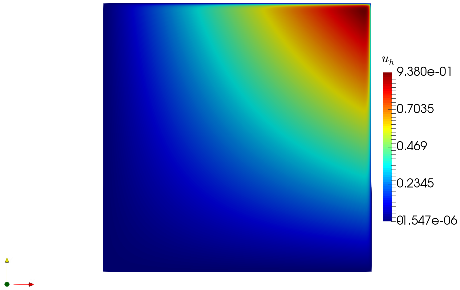
$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = s(\mathbf{x}) \qquad \mathbf{x} \in \Omega = (0,1)^2$$
$$u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \partial\Omega$$

where $\boldsymbol{\beta}$ is $[1,1]^T$.

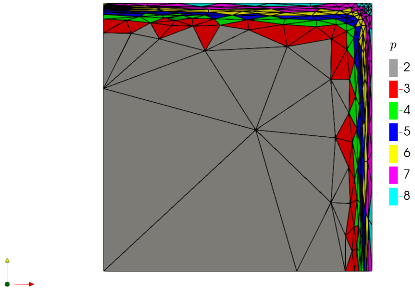
- Source term $s(\mathbf{x})$ is chosen in such a way that the exact solution is given by:

$$u(\mathbf{x}) = \prod_{i=1}^{2} \left(x_i + \frac{e^{\frac{x_i}{\epsilon}} - 1}{1 - e^{\frac{1}{\epsilon}}} \right)$$

Scalar Boundary Layer



(a)



(b)

Figure: Boundary layer: (a) solution contour on an adapted mesh and (b) polynomial distribution on the same adapted mesh with $\epsilon = 0.005$ with 6980 degrees of freedom.

Scalar Boundary Layer

Adaptation	Ne	$\|u - u_h\|_{2,\Omega}$	$\|U - U_h\|_{E,\Omega}$
0	512	0.0262192	0.0514999
2	437	0.000289966	0.00218748
4	463	6.05824e-06	4.77721e-05
6	484	3.65609e-06	3.09974e-05
8	489	2.02129e-06	1.86422e-05

Table: Adaptation Vs. error for constant complexity using scaled V-norm ($\epsilon = 0.005$, $p = 3$, $N = \int_\Omega d(\mathbf{x})\, d\mathbf{x} = 512.0 \times \frac{3\sqrt{3}}{4}$).

Scalar Boundary Layer

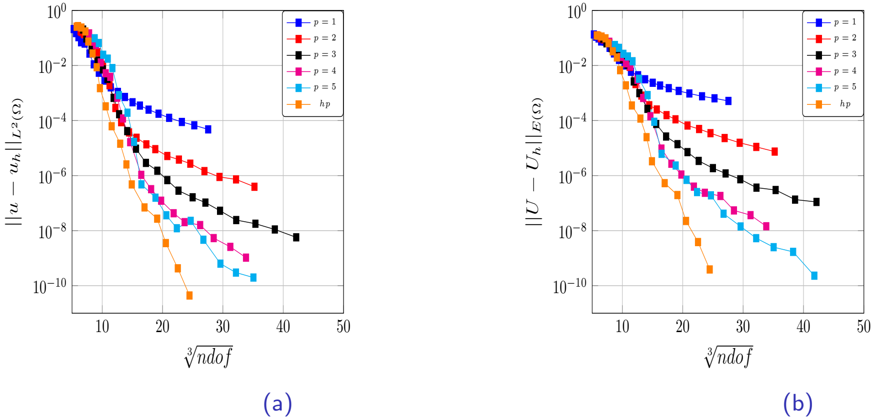
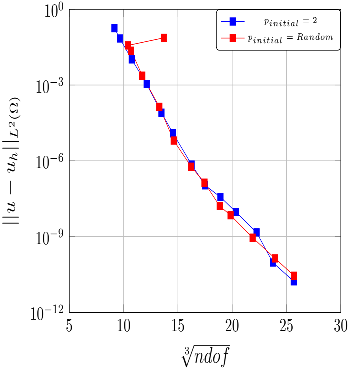
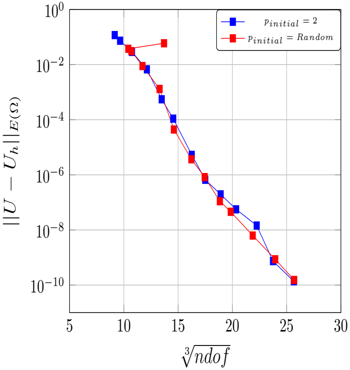


Figure: Convergence plots of (a) L^2 error in u_h and (b) energy norm using scaled V-norm with $\epsilon = 0.005$.

Scalar Boundary Layer



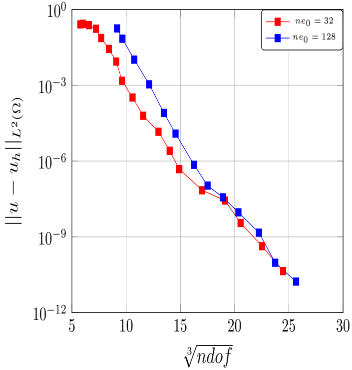
(a)



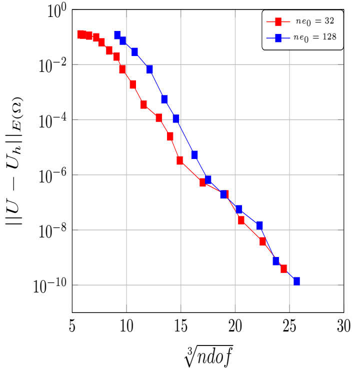
(b)

Figure: Convergence plots of (a) L^2 error in u_h and (b) energy norm using scaled V-norm with different initial polynomial distributions ($\epsilon = 0.005$).

Scalar Boundary Layer



(a)



(b)

Figure: Convergence plots of (a) L^2 error in u_h and (b) energy norm using scaled V-norm with different initial mesh ($\epsilon = 0.005$).

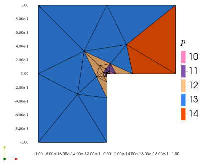
L-Shaped Domain

$$-\nabla^2 u = s(\mathbf{x}) \qquad \mathbf{x} \in \Omega = [-1,1]^2 \setminus [0,1] \times [-1,0]$$
$$u = g_D \qquad \mathbf{x} \in \partial\Omega$$

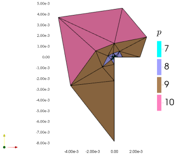
Source term $s(\mathbf{x})$ is chosen in such a way that the exact solution is given by:

$$u(\mathbf{x}) = r^{\frac{2}{3}} sin\left(\frac{2}{3}\theta\right) \quad where \quad \theta = tan^{-1}\left(\frac{x_2}{x_1}\right)$$

L-Shaped Domain



(a)



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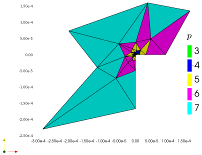
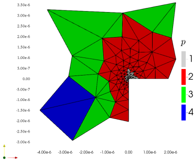


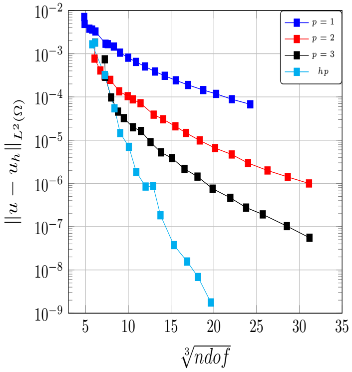
(c)



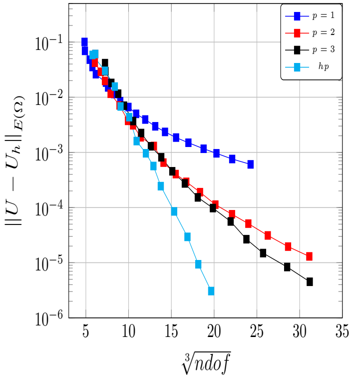
(d)

Figure: Polynomial Distribution

L-Shaped Domain



(a)



(b)

Figure: Convergence plots of (a) L^2 error in u_h and (b) energy norm using scaled V-norm.

Goal Oriented Adaptation : Scalar Boundary Layer

Viscous linear equation with source

$$\begin{aligned} \nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u &= s_1 \qquad \boldsymbol{x} \in \Omega = (0,1)^2 \\ u(\boldsymbol{x}) &= 0 \qquad \boldsymbol{x} \in \partial\Omega \end{aligned}$$

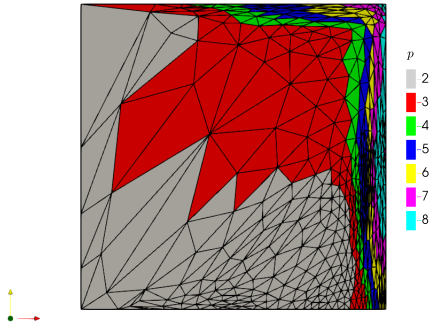
where $\boldsymbol{\beta}$ is $[1,1]^T$. Source term $s(\boldsymbol{x})$ is chosen in such a way that the exact solution is given by:

$$u(\boldsymbol{x}) = \prod_{i=1}^{2} \left(x_i + \frac{e^{\frac{x_i}{\epsilon}} - 1}{1 - e^{\frac{1}{\epsilon}}} \right)$$

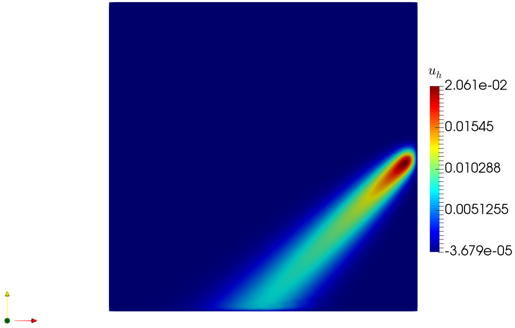
$$J(u) = \int_\Omega e^{-\alpha r^2} u(\boldsymbol{x})\, d\boldsymbol{x} \quad where \quad \alpha = 1000$$

$$r = \sqrt{(x - x_c)^2 + (y - y_c)^2} \quad where \quad (x_c, y_c) = (0.99, 0.5)$$

Goal Oriented Adaptation : Scalar Boundary Layer



(a)



(b)

Figure: Contour showing (a) Polynomial distribution and (b)Dual solution

Goal Oriented Adaptation : Scalar Boundary Layer

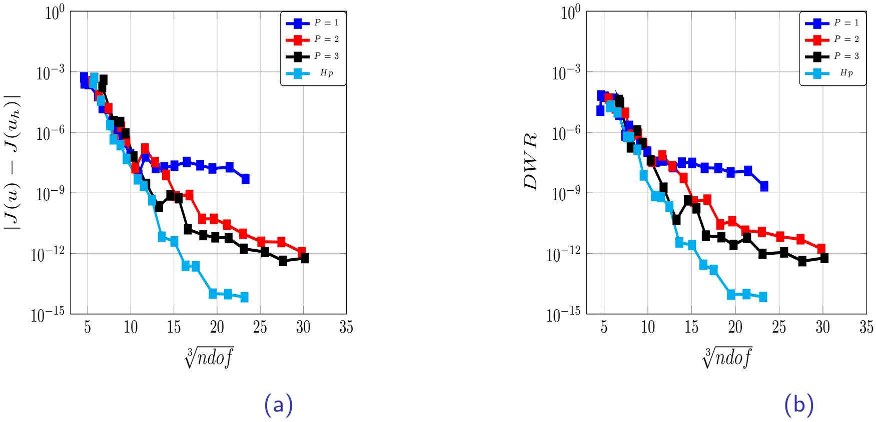


Figure: Convergence plots for (a) error in target functional and (b) dual Weighted Residual using scaled V norm.

Goal Oriented Adaptation : Scalar Boundary Layer

Adaptation	Ndof	$\lvert J(u) - J(u_h) \rvert$	DWR
0	3072	1.3599e-04	2.55956e-05
2	2698	2.62413e-09	1.16551e-09
4	3271	1.2955e-11	1.69234e-11
6	3045	2.92165e-12	2.71505e-12
8	3276	3.98067e-14	1.31114e-13

Table: Adaptation Vs. Error for constant complexity using scaled V norm. $(N = \int_\Omega w(x)d(x)\,dx = 512.0)$

Goal Oriented Adaptation: Double Inverse Tangent

The source term $s(\mathbf{x})$ is selected in such a way that the exact solution is given by

$$u(\mathbf{x}) = \left(tan^{-1}(\alpha(x - x_1)) + tan^{-1}(\alpha(x_2 - x))\right)\left(tan^{-1}(\alpha(y - y_1)) + tan^{-1}(\alpha(y_2 - y))\right)$$

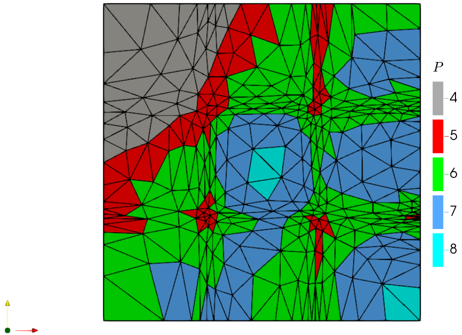
where $x_1 = y_1 = \frac{1.0}{3.0}$, $x_2 = y_2 = \frac{2.0}{3.0}$, $\alpha = 50.0$ and $\epsilon = 0.01$. The target for this problem is

$$J(\sigma) = \int_{\partial\Omega} j_{\partial\Omega}(\boldsymbol{x})\boldsymbol{\sigma} \cdot \mathbf{n} ds$$

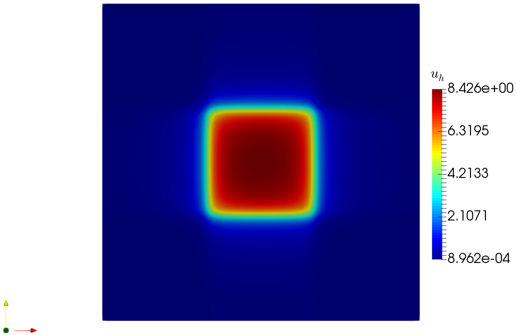
where we have

$$j_{\partial\Omega} = \begin{cases} 1 & \mathbf{n} = (1, 0) \\ 0 & otherwise \end{cases}$$

Goal Oriented Adaptation: Double Inverse Tangent



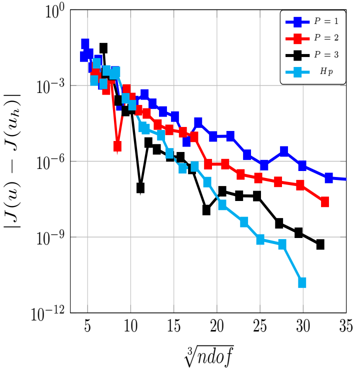
(a)



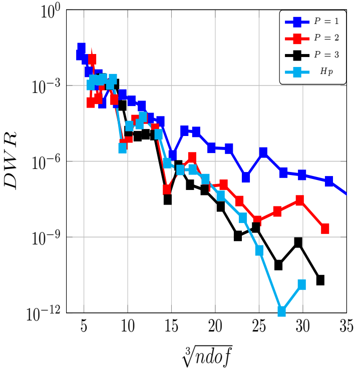
(b)

Figure: Contour showing (a) Polynomial distribution and (b)Primal solution

Goal Oriented Adaptation: Double Inverse Tangent



(a)



(b)

Figure: Convergence plots for (a) error in target functional and (b) dual Weighted Residual using scaled V norm.

Conclusions

- Based on a robust inbuilt error estimate.
- Efficient in terms of degrees of freedom required to achieve the same level of error.
- Analytic optimization for global size distribution.

Future Work

- Reduce the cost of anisotropy computations.

[1] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov – Galerkin methods. II. optimal test functions. *Numerical Methods for Partial Differential Equations*, 27(1):70–105, 2011. doi: 10.1002/num.20640.

[2] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. Part I: The transport equation. *Computer Methods in Applied Mechanics and Engineering*, 199(23-24): 1558–1572, 2010. ISSN 00457825. doi: 10.1016/j.cma.2010.01.003.

[3] L. Demkowicz and N. Heuer. Robust dpg method for convection-dominated diffusion problems. *SIAM Journal on Numerical Analysis*, 51(5):2514–2537, 2013. doi: 10.1137/120862065.

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Financial support from the
Deutsche Forschungsgemeinschaft (DFG)
through grant IRTG-2379 is gratefully acknowledged