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Space-time discontinuous Galerkin methods and discontinuous Petrov-Galerkin methods for hyperbolic linear symmetric Friedrichs systems

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The linear acoustic wave equation

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Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in space with Lipschitz boundary,
$I = (0, T)$ a time interval, and $Q = (0, T) \times \Omega$ the space-time cylinder.
The second-order wave equation

$$\varrho \partial_t^2 \phi - \nabla \cdot (\kappa \nabla \phi) = b$$

is considered as first-order system with $p = \partial_t \phi$ and $\mathbf{q} = -\kappa \nabla \phi$, i.e.,

$$\begin{aligned} \varrho \partial_t p + \nabla \cdot \mathbf{q} = b \ \text{ and } \ \partial_t \mathbf{q} + \kappa \nabla p = \mathbf{0} \quad & \text{in } (0, T) \times \Omega \,, \\ p(0) = p_0 \ \text{ and } \ \mathbf{q}(0) = \mathbf{q}_0 \quad & \text{in } \Omega \text{ at } t = 0 \,, \\ p(t) = p_{\mathrm{D}}(t) \text{ on } \Gamma_{\mathrm{D}} \ \text{ and } \ \mathbf{n} \cdot \mathbf{q}(t) = g_{\mathrm{N}}(t) \text{ on } \Gamma_{\mathrm{N}} \quad & \text{on } \partial\Omega \text{ for } t \in (0, T) \end{aligned}$$

with volume data b, boundary data g_{N}, p_{D}, initial data $\mathbf{q}_0$, p_0, uniformly positive parameters ϱ, κ, and the disjoint decomposition of the boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$.

$(p, \mathbf{q}) \in \mathrm{L}_2(Q; \mathbb{R}^{1+d})$ is a *weak solution*, if

$$\begin{aligned} -\big(p, \varrho \partial_t \varphi\big)_Q - \big(\mathbf{q}, \kappa^{-1} \partial_t \boldsymbol{\eta}\big)_Q - \big(p, \nabla \cdot \boldsymbol{\eta}\big)_Q - \big(\mathbf{q}, \nabla p\big)_Q & \\ = \big(b, \varphi\big)_Q + \big(p_0, \varphi(0)\big)_\Omega + \big(\mathbf{q}_0, \boldsymbol{\eta}(0)\big)_\Omega - \big(p_{\mathrm{D}}, \mathbf{n} \cdot \boldsymbol{\eta}\big)_{(0,T)\times\Gamma_{\mathrm{D}}} & - \big(g_{\mathrm{N}}, \varphi\big)_{(0,T)\times\Gamma_{\mathrm{N}}} \end{aligned}$$

for smooth test functions $(\varphi, \boldsymbol{\eta})$ with $\varphi(T) = 0$ and $\boldsymbol{\eta}(T) = \mathbf{0}$.

The linear acoustic wave equation

The corresponding Friedrichs system with $m = 1 + d$ components is given by

$$\mathbf{u} = \begin{pmatrix} p \\ \mathbf{q} \end{pmatrix}, \ M\mathbf{u} = \begin{pmatrix} \varrho p \\ \kappa^{-1}\mathbf{q} \end{pmatrix}, \ A\mathbf{u} = \begin{pmatrix} \nabla\cdot\mathbf{q} \\ \nabla p \end{pmatrix}, \ \underline{A}_{\mathbf{n}}\mathbf{u} = \begin{pmatrix} \mathbf{n}\cdot\mathbf{q} \\ p\mathbf{n} \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix}, \ \mathbf{g} = \begin{pmatrix} g_{\mathrm{N}} \\ p_{\mathrm{D}}\mathbf{n} \end{pmatrix}$$

and $L\mathbf{u} = M\partial_t\mathbf{u} + A\mathbf{u}$, $L^* = -L$, so that for test functions $\mathbf{w} \in \mathcal{V}^*$

$$\big(\mathbf{f}, \mathbf{w}\big)_Q = \big(L\mathbf{u}, \mathbf{w}\big)_Q = \big(\mathbf{u}, L^*\mathbf{w}\big)_Q - \big(\mathbf{u}_0, \mathbf{w}(0)\big)_\Omega + \big(\mathbf{g}, \mathbf{w}\big)_{(0,T)\times\partial\Omega}$$

with $\mathcal{V}^* = \big\{\mathbf{w} = (\varphi, \boldsymbol{\eta}) \in \mathrm{C}^1(\overline{Q}; \mathbb{R}^m) \colon \mathbf{w}(T) = 0 \text{ in } \Omega\,,\ \varphi = 0 \text{ on } (0, T) \times \Gamma_{\mathrm{D}}\,,$
$\mathbf{n}\cdot\boldsymbol{\eta} = 0 \text{ on } (0, T) \times \Gamma_{\mathrm{N}}\big\}\,.$

Our aim is to find a *weak solution* $\mathbf{u} \in \mathrm{L}_2(Q; \mathbb{R}^m)$ solving for all $\mathbf{w} \in \mathcal{V}^*$

$$\big(\mathbf{u}, L^*\mathbf{w}\big)_Q = \langle \ell, \mathbf{w}\rangle \quad \text{with} \quad \langle \ell, \mathbf{w}\rangle = \big(\mathbf{f}, \mathbf{w}\big)_Q + \big(M\mathbf{u}_0, \mathbf{w}(0)\big)_\Omega - \big(\mathbf{g}, \mathbf{w}\big)_{(0,T)\times\partial\Omega}$$

for given volume data $\mathbf{f} \in \mathrm{L}_2(Q; \mathbb{R}^m)$, initial data $\mathbf{u}_0 \in \mathrm{L}_2(\Omega; \mathbb{R}^m)$, and boundary data $\mathbf{g} \in \mathrm{L}_2((0, T) \times \partial\Omega; \mathbb{R}^m)$, where the boundary data $\mathbf{g} = (g_k)_{k=1,\dots,m}$ are extended to $\partial\Omega$ by $g_k = 0$ on $\partial\Omega \setminus \Gamma_k$ for $k = 1, \dots, m$.

The weak solution is also a *strong solution* if

$$L\mathbf{u} = \mathbf{f} \text{ in } \mathrm{L}_2(Q; \mathbb{R}^m)\,,\ \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \mathrm{L}_2(\Omega; \mathbb{R}^m)\,,\ (\underline{A}_{\mathbf{n}}\mathbf{u})_k = g_k \text{ on } \mathrm{L}_2((0, T) \times \Gamma_k)\,,\ \forall k\,.$$

Existence of a weak solution

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Lemma

$\|M^{1/2}\mathbf{w}\|_Q \le C_L^* \|M^{-1/2}L^*\mathbf{w}\|_Q$ *for* $\mathbf{w} \in \mathcal{V}^*$ *holds with* $C_L^* = 2T$.

Define $Z = \big\{\mathbf{w} = (\varphi, \boldsymbol{\eta}) \in \mathrm{H}^1(\Omega) \times \mathrm{H}(\mathrm{div}, \Omega) \colon \varphi = 0 \text{ on } \Gamma_{\mathrm{D}}\,,\ \mathbf{n}\cdot\boldsymbol{\eta} = 0 \text{ on } \Gamma_{\mathrm{N}}\big\}$.

Lemma

Assume that $M + \tau A \colon Z \to \mathrm{L}_2(\Omega; \mathbb{R}^m)$ *is surjective for all* $\tau > 0$.
Then, $L^*(\mathcal{V}^*) \subset \mathrm{L}_2(Q; \mathbb{R}^m)$ *is dense.*

This is proved by time stepping and holds for our application to acoustics.

We define the quadratic functional $J^*(\mathbf{w}) = \frac{1}{2}\|M^{-1/2}L^*\mathbf{w}\|_Q^2 - \langle \ell, \mathbf{w}\rangle$ for $\mathbf{w} \in \mathcal{V}^*$.
Let V^* be the closure of $\mathcal{V}^*$ in $\big\{\mathbf{v} \in \mathrm{L}_2(Q; \mathbb{R}^m) \colon L\mathbf{v} \in \mathrm{L}_2(Q; \mathbb{R}^m)\big\}$.

Theorem

Assume that $C_\ell > 0$ *exists with* $|\langle \ell, \mathbf{w}\rangle| \le C_\ell \|M^{-1/2}L^*\mathbf{w}\|_Q$ *for* $\mathbf{w} \in \mathcal{V}^*$.

Then, a unique minimizer $\mathbf{z} \in V^*$ *of* $J^*(\cdot)$ *exists, and* $\mathbf{u} = L^*\mathbf{z} \in \mathrm{L}_2(Q; \mathbb{R}^m)$ *is the unique weak solution of*

$$(\mathbf{u}, L^*\mathbf{w})_Q = \langle \ell, \mathbf{w}\rangle\,, \qquad \mathbf{w} \in \mathcal{V}^*\,.$$

The DG finite element space in the space-time cylinder

For $0 = t_0 < t_1 < \cdots < t_N = T$, we define time intervals $I_{n,h} = (t_{n-1}, t_n)$ and

$$I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)\,, \qquad \partial I_h = \{t_0, t_1, \dots, t_{N-1}, t_N\}\,.$$

Let $\mathcal{K}_h$ be a mesh so that $\Omega_h = \bigcup_{K \in \mathcal{K}_h} K$ is a decomposition in space into open cells $K \subset \Omega \subset \mathbb{R}^d$. We obtain a decomposition into $R = I_{n,h} \times K$ and

$$Q_h = I_h \times \Omega_h = \bigcup_{n=1}^{N} Q_{n,h} = \bigcup_{R \in \mathcal{R}_h} R\,, \qquad Q_{n,h} = \bigcup_{K \in \mathcal{K}_h} I_{n,h} \times K \ \subset \ I_{n,h} \times \Omega\,.$$

We select a finite dimensional subspace $V_h \subset \mathcal{V}_h \subset \mathrm{C}^1(Q_h; \mathbb{R}^m)$, where

$$\mathcal{V}_h = \big\{ \mathbf{v}_h \in \mathrm{C}^1(Q_h; \mathbb{R}^m) \colon \mathbf{v}_{n,h,K} = \mathbf{v}_h|_{I_{n,h} \times K} \text{ extends to } \mathbf{v}_{n,h,K} \in \mathrm{C}^0(\overline{I_{n,h} \times K}; \mathbb{R}^m) \big\}\,.$$

Let $F \in \mathcal{F}_K$ be the faces of the element K, and we set $\mathcal{F}_h = \bigcup_K \mathcal{F}_K$.

We define the time-step sizes $\triangle t_n = t_n - t_{n-1}$ and $\triangle t = \max \triangle t_n$.

We set $h_K = \operatorname{diam} K$, $h_F = \operatorname{diam} F$, and $h = \max h_K$, and we assume $h_F \geq c_{\mathrm{sr}} h_K$.

We assume quasi-uniform meshes and shape-regularity.

We assume, depending on a reference velocity $c_{\mathrm{ref}} > 0$,
that the mesh size in time and space are well balanced satisfying $c_{\mathrm{ref}} \triangle t \leq h$.

The DG discretization

Let M_h be a piecewise constant approximation of M, define $L_h \mathbf{v}_h = M_h \partial_t \mathbf{v}_h + A\mathbf{v}_h$.

For $\mathbf{v}_h = (p_h, \mathbf{q}_h) \in \mathcal{V}_h$ and $\mathbf{w}_h = (\varphi_h, \boldsymbol{\eta}_h) \in \mathcal{V}_h$ we have

$$\big(L_h \mathbf{v}_h, \mathbf{w}_h\big)_{Q_h} = -\big(\mathbf{v}_h, L_h \mathbf{w}_h\big)_{Q_h} + \sum_{n=1}^{N} \big(M_h \mathbf{v}_{n,h}(t_n), \mathbf{w}_{n,h}(t_n)\big)_\Omega - \big(M_h \mathbf{v}_{n,h}(t_{n-1}), \mathbf{w}_{n,h}(t_{n-1})\big)_\Omega$$
$$+ \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K} \mathbf{v}_{h,K}, \mathbf{w}_h\big)_{I_h \times F} .$$

In the DPG method, we introduce additional trace degrees of freedom on

- the interfaces in time $\{t_n\} \times \Omega$ for $\mathbf{v}_{n,h}(t_n)$
- on the faces in space $(t_{n-1}, t_n) \times F$ for $\underline{A}_{\mathbf{n}_F} \mathbf{v}_{h,K}$ selecting $\mathbf{n}_F \in \pm \mathbf{n}_K$

The DG discretization with full upwind

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Let M_h be a piecewise constant approximation of M, define $L_h \mathbf{v}_h = M_h \partial_t \mathbf{v}_h + A\mathbf{v}_h$.

For $\mathbf{v}_h = (p_h, \mathbf{q}_h) \in \mathcal{V}_h$ and $\mathbf{w}_h = (\varphi_h, \boldsymbol{\eta}_h) \in \mathcal{V}_h$ we define the discrete bilinear form

$$b_h(\mathbf{v}_h, \mathbf{w}_h) = -\big(\mathbf{v}_h, L_h \mathbf{w}_h\big)_{Q_h} - \sum_{n=1}^{N} \big(M_h \mathbf{v}_{n,h}(t_n), [\mathbf{w}_h]_n\big)_\Omega + \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \big(\mathbf{v}_{h,K}, \underline{A}^{\mathrm{up}}_{\mathbf{n}_K} [\mathbf{w}_h]_{K,F}\big)_{I_h \times F},$$

where the upwind flux $\underline{A}^{\mathrm{up}}_{\mathbf{n}_K} \in \mathbb{R}^{m \times m}_{\mathrm{sym}}$ is obtained by solving local Riemann problems, and defining $[\mathbf{w}_h]_n = \mathbf{w}_{n+1,h}(t_n) - \mathbf{w}_{n,h}(t_n)$ for $n = 1, \dots, N-1$, $[\mathbf{w}_h]_N = -\mathbf{w}_{N,h}(t_N)$.
On inner faces $F \subset \Omega$, we define the jump term $[\mathbf{w}_h]_{K,F} = \mathbf{w}_{h,K_F} - \mathbf{w}_{h,K}$.
On boundary faces $F \subset \partial\Omega$ this depends on the boundary conditions, and we set $(\underline{A}_{\mathbf{n}}[\mathbf{v}_h])_k = -2(\underline{A}_{\mathbf{n}} \mathbf{v}_h)_k$ on $\Gamma_k \subset \partial\Omega$ and $(\underline{A}_{\mathbf{n}}[\mathbf{v}_h])_k = 0$ on $\partial\Omega \setminus \Gamma_k$ for $k = 1, \dots, m$.

For acoustics we have, depending on the impedance $Z_K = \sqrt{\kappa_{h,K} \varrho_{h,K}}$,

$$\big(\mathbf{v}_{h,K}, \underline{A}^{\mathrm{up}}_{\mathbf{n}_K} [\mathbf{w}_h]_{K,F}\big)_F = -\frac{1}{Z_K + Z_{K_F}} \big(p_{K,h} + Z_{K_F} \mathbf{n}_K \cdot \mathbf{q}_{K,h}, [\varphi_h]_{K,F} + Z_K \mathbf{n}_K \cdot [\boldsymbol{\psi}_h]_{K,F}\big)_F,$$

$$\big\langle \ell_h, (\varphi_h, \boldsymbol{\eta}_h) \big\rangle = \big(b, \varphi_h\big)_Q + \big(p_0, \varphi_h(0)\big)_\Omega + \big(\mathbf{q}_0, \boldsymbol{\eta}_h(0)\big)_\Omega - \big(p_{\mathrm{D}}, \mathbf{n} \cdot \boldsymbol{\eta}_h - Z_h^{-1} \varphi_h\big)_{(0,T) \times \Gamma_{\mathrm{D}}} - \big(g_{\mathrm{D}}, \varphi_h - Z_h \mathbf{n} \cdot \boldsymbol{\psi}_h\big)_{(0,T) \times \Gamma_{\mathrm{N}}}.$$

Consistency of the DG discretization

For $\mathbf{v}_h, \mathbf{w}_h \in \mathcal{V}_h$ we have

$$b_h(\mathbf{v}_h, \mathbf{v}_h) \geq \frac{1}{2} \sum_{n=0}^{N} \big\| M_h^{1/2} [\mathbf{v}_h]_n \big\|_\Omega^2 + c_1 \big\| \underline{A}_\mathbf{n} [\mathbf{v}_h] \big\|_{I_h \times \partial\Omega_h}^2$$

$$b_h(\mathbf{v}_h, \mathbf{v}_h) = 0 \quad \Longrightarrow \quad b_h(\mathbf{v}_h, \mathbf{w}_h) = \big(\mathbf{v}_h, L_h^* \mathbf{w}_h\big)_{Q_h} = \big(L_h \mathbf{v}_h, \mathbf{w}_h\big)_{Q_h}$$

$$\big| b_h(\mathbf{v}_h, \mathbf{w}_h) - \big(\mathbf{v}_h, L_h^* \mathbf{w}_h\big)_{\Omega_h} \big| \leq \big\| M_h^{1/2} \mathbf{v}_h \big\|_{\partial Q_h} \sqrt{\big\| M_h^{1/2} [\mathbf{w}_h] \big\|_{\partial I_h \times \Omega}^2 + C_1 \big\| \underline{A}_\mathbf{n} [\mathbf{w}_h] \big\|_{I_h \times \partial\Omega_h}^2}$$

$$\big| b_h(\mathbf{v}_h, \mathbf{w}_h) - \big(L_h \mathbf{v}_h, \mathbf{w}_h\big)_{\Omega_h} \big| \leq \sqrt{\big\| M_h^{1/2} [\mathbf{v}_h] \big\|_{\partial I_h \times \Omega}^2 + C_1 \big\| \underline{A}_\mathbf{n} [\mathbf{v}_h] \big\|_{I_h \times \partial\Omega_h}^2} \; \big\| M_h^{1/2} \mathbf{w}_h \big\|_{\partial Q_h}$$

The mesh-dependent DG semi-norm and norm is defined for $\mathbf{v}_h \in \mathcal{V}_h$ by

$$\big| \mathbf{v}_h \big|_{h,\mathrm{DG}} = \sqrt{b_h(\mathbf{v}_h, \mathbf{v}_h)}\,, \qquad \big\| \mathbf{v}_h \big\|_{h,\mathrm{DG}} = \sqrt{\big| \mathbf{v}_h \big|_{h,\mathrm{DG}}^2 + \big\| h\, M_h^{-1/2} L_h \mathbf{v}_h \big\|_{Q_h}^2}\,.$$

$$\big| \mathbf{v}_h \big|_{h,\mathrm{DG}^+} = \sqrt{\sum_{n=1}^{N} \Big(\big\| M_h^{1/2} \mathbf{v}_{n,h}(t_{n-1}) \big\|_\Omega^2 + \big\| M_h^{1/2} \mathbf{v}_{n,h}(t_n) \big\|_\Omega^2 \Big) + C_1 \sum_{K \in \mathcal{K}_h} \big\| M_h^{1/2} \mathbf{v}_h \big\|_{I_h \times \partial K}^2}$$

and $\big\| \mathbf{v}_h \big\|_{h,\mathrm{DG}^+} = \sqrt{\big| \mathbf{v}_h \big|_{h,\mathrm{DG}^+}^2 + \big\| h^{-1/2} M_h^{1/2} \mathbf{v}_h \big\|_Q^2}$ yields

$$b_h(\mathbf{v}_h, \mathbf{w}_h) \leq \big\| \mathbf{v}_h \big\|_{h,\mathrm{DG}} \big\| \mathbf{w}_h \big\|_{h,\mathrm{DG}^+} \quad \text{and} \quad b_h(\mathbf{v}_h, \mathbf{w}_h) \leq \big\| \mathbf{v}_h \big\|_{h,\mathrm{DG}^+} \big\| \mathbf{w}_h \big\|_{h,\mathrm{DG}}\,.$$

Well-posedness and stability

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Lemma

Define $d_T(t) = T - t$. We have

$$\big\|M_h^{1/2}\mathbf{v}_h\big\|_Q^2 + T\,\big\|M_h^{1/2}\mathbf{v}_h(0)\big\|_\Omega^2 \le 2\ b_h(\mathbf{v}_h, d_T\mathbf{v}_h)\,, \qquad \mathbf{v}_h \in \mathcal{V}_h\,.$$

Let $V_h \subset \mathbb{P}(Q_h;\mathbb{R}^m) \subset \mathcal{V}_h$ be a finite dimensional subspace.

Lemma

A unique discrete approximation $\mathbf{u}_h \in V_h$ exists solving

$$b_h(\mathbf{u}_h, \mathbf{v}_h) = \big\langle \ell_h, \mathbf{v}_h\big\rangle\,, \qquad \mathbf{v}_h \in V_h\,.$$

Depending on $\big\|h^{1/2}M_h^{-1/2}L_h\mathbf{v}_h\big\|_{Q_h} \le C_{\text{inv}}\,\big\|h^{-1/2}M_h^{1/2}\mathbf{v}_h\big\|_Q$ and $\big\|M_h^{1/2}\mathbf{v}_h\big\|_{\partial Q_h} \le C_{\text{tr}}\,\big\|h^{-1/2}M_h^{1/2}\mathbf{v}_h\big\|_Q$ for $\mathbf{v}_h \in V_h$ we obtain

Theorem

$$\sup_{\mathbf{w}_h\in V_h\setminus\{\mathbf{0}\}} \frac{b_h(\mathbf{v}_h,\mathbf{w}_h)}{\big\|\mathbf{w}_h\big\|_{h,\mathrm{DG}}} \ge c_{\text{inf-sup}}\big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}, \quad \mathbf{v}_h \in V_h \text{ with } c_{\text{inf-sup}} = \frac{1}{4 + 2C_{tr}^2 + 2C_{inv}}.$$

Explicit construction of the inf-sup estimate

For given $\mathbf{v}_h \in V_h \setminus \{\mathbf{0}\}$ we define $\mathbf{z}_h = hM_h^{-1} L_h \mathbf{v}_h \in V_h$, and we obtain

$$\big|\mathbf{z}_h\big|_{h,\mathrm{DG}^+} \leq C_{\mathrm{tr}} \big\| h^{-1/2} M_h^{1/2} \mathbf{z}_h \big\|_{Q_h} = C_{\mathrm{tr}} \big\| h^{1/2} M_h^{-1/2} L_h \mathbf{v}_h \big\|_{Q_h} \leq C_{\mathrm{tr}} \big\| \mathbf{v}_h \big\|_{h,\mathrm{DG}} .$$

Together with the inverse inequality this yields $\big\|\mathbf{z}_h\big\|_{h,\mathrm{DG}}^2 \leq \big(C_{\mathrm{tr}}^2 + C_{\mathrm{inv}}^2\big) \big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}^2$.
We observe

$$\big(L_h \mathbf{v}_h, \mathbf{z}_h\big)_{Q_h} - b_h(\mathbf{v}_h, \mathbf{z}_h) \leq \big|\mathbf{v}_h\big|_{h,\mathrm{DG}} \big|\mathbf{z}_h\big|_{h,\mathrm{DG}^+} \leq \frac{C_{\mathrm{tr}}^2}{2} \big|\mathbf{v}_h\big|_{h,\mathrm{DG}}^2 + \frac{1}{2} \big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}^2 .$$

This yields

$$\big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}^2 \leq \big|\mathbf{v}_h\big|_{h,\mathrm{DG}}^2 + \frac{C_{\mathrm{tr}}^2}{2} \big|\mathbf{v}_h\big|_{h,\mathrm{DG}}^2 + \frac{1}{2} \big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}^2 + b_h(\mathbf{v}_h, \mathbf{z}_h) ,$$

so that with $C_2 = 2 + C_{\mathrm{tr}}^2$

$$\big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}^2 \leq C_2 \big|\mathbf{v}_h\big|_{h,\mathrm{DG}}^2 + 2\, b_h(\mathbf{v}_h, \mathbf{z}_h) = b_h(\mathbf{v}_h, C_2 \mathbf{v}_h + 2\mathbf{z}_h) .$$

We obtain the assertion with $\mathbf{y}_h = C_2 \mathbf{v}_h + 2\mathbf{z}_h$ by

$$\big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}}^2 \leq \big\|\mathbf{y}_h\big\|_{h,\mathrm{DG}} \frac{b_h(\mathbf{v}_h, \mathbf{y}_h)}{\big\|\mathbf{y}_h\big\|_{h,\mathrm{DG}}} \leq c_{\mathrm{inf\text{-}sup}}^{-1} \big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}} \sup_{\mathbf{w}_h \in V_h \setminus \{\mathbf{0}\}} \frac{b_h(\mathbf{v}_h, \mathbf{w}_h)}{\big\|\mathbf{w}_h\big\|_{h,\mathrm{DG}}} .$$

Convergence in the DG norm

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Theorem

Assume that the solution is sufficiently smooth satisfying $\mathbf{u} \in \mathrm{H}^s(Q;\mathbb{R}^m)$ *with* $s \geq 1$*. Then, the error for the discrete solution* $\mathbf{u}_h \in V_h$ *is bounded by*

$$\big\|\mathbf{u} - \mathbf{u}_h\big\|_{h,\mathrm{DG}} \leq Ch^{s-1/2}\,\big\|\mathrm{D}^s\mathbf{u}\big\|_Q + CT\,h^{-1/2}\big\|M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\big\|_Q\,.$$

$C > 0$ depends on mesh regularity / polynomial degree / material parameters.

Corollary

$$\big\|\mathbf{u}(t_n) - \mathbf{u}_{n,h}(t_n)\big\|_\Omega \leq Ch^{s-1/2}\big\|\mathrm{D}^s\mathbf{u}\big\|_{(0,t_n)\times\Omega} + C\,t_n\,h^{-1/2}\big\|M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\big\|_{(0,t_n)\times\Omega}$$

If M is discontinuous and the parameters ϱ and κ are not resolved by the mesh, the consistency error can be estimated in case of higher regularity:

$$\big\|\big(M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\big\|_Q \leq \big\|M_h^{-1/2}(M - M_h)M^{-1/2}\big\|_{\mathrm{L}_{2q/(2-q)}(\Omega)}\big\|M^{1/2}\partial_t\mathbf{u}\big\|_{\mathrm{L}_2(0,T;\mathrm{L}_q(\Omega))}$$

in case of $\partial_t\mathbf{u} \in \mathrm{L}_2(0,T;\mathrm{L}_q(\Omega;\mathbb{R}^m))$ with $q > 2$.

The direct minimal residual approach

The DG solution $\mathbf{u}_h \in V_h$ is the unique minimizer of the functional

$$J_{h,\mathrm{DG}}(\mathbf{v}_h) = \sup_{\mathbf{w}_h \in V_h \setminus \{\mathbf{0}\}} \frac{\big| b_h(\mathbf{v}_h, \mathbf{w}_h) - \langle \ell_h, \mathbf{w}_h \rangle \big|}{\big\| \mathbf{w}_h \big\|_{h,\mathrm{DG}}}, \qquad \mathbf{v}_h \in V_h\,.$$

with $J_{h,\mathrm{DG}}(\mathbf{u}_h) = 0$.
The solution can be obtained by solving a symmetric positive definite Schur complement problem. The Schur complement matrix cannot be computed locally.

The direct minimal residual approach

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The DG solution $\mathbf{u}_h \in V_h$ is the unique minimizer of the functional

$$J_{h,\mathrm{DG}^+}(\mathbf{v}_h) = \sup_{\mathbf{w}_h \in V_h \setminus \{\mathbf{0}\}} \frac{\left| b_h(\mathbf{v}_h, \mathbf{w}_h) - \langle \ell_h, \mathbf{w}_h \rangle \right|}{\left\| \mathbf{w}_h \right\|_{h,\mathrm{DG}^+}}\,, \qquad \mathbf{v}_h \in V_h$$

with $J_{h,\mathrm{DG}^+}(\mathbf{u}_h) = 0$.
The solution can be obtained by solving a symmetric positive definite Schur complement problem. The Schur complement matrix can be computed locally.

Therefore, define operators M_{n,K,DG^+} and $B_{n,K}$ with

$$\left\| \mathbf{w}_h \right\|^2_{h,\mathrm{DG}^+} = \sum_n \sum_K \left(M_{n,K,\mathrm{DG}^+} \mathbf{w}_h, \mathbf{w}_h \right)_{I_{n,h} \times K}\,,$$

$$b_h(\mathbf{v}_h, \mathbf{w}_h) = \sum_n \sum_K \left(B_{n,K} \mathbf{w}_h, \mathbf{w}_h \right)_{I_{n,h} \times K}\,,$$

$$\left(B_{n,K} \mathbf{v}_h, \mathbf{w}_h \right)_{I_{n,h} \times K} = -\left(\mathbf{v}_h, L_h \mathbf{w}_h \right)_{I_{n,h} \times K} - \left(M_h \mathbf{v}_{n,h}(t_n), [\mathbf{w}_h]_n \right)_\Omega + \sum_{F \in \mathcal{F}_K} \left(\mathbf{v}_{h,K}, \underline{A}^{\mathrm{up}}_{\mathbf{n}_K} [\mathbf{w}_h]_{K,F} \right)_{I_{n,h} \times F}\,,$$

so that

$$S_h \mathbf{u}_h = \mathbf{f}_h \text{ with } S_h = \sum_n \sum_K B^\top_{n,K} M^{-1}_{n,K,\mathrm{DG}^+} B_{n,K}\,, \quad \left(\mathbf{f}_h, \mathbf{w}_h \right)_Q = \langle \ell_h, \mathbf{w}_h \rangle\,.$$

The ideal minimal residual approach

A minimizer $\mathbf{u}_h^{\text{exact}} \in V_h$ of

$$J(\mathbf{v}_h) = \sup_{\mathbf{w}\in\mathcal{V}^*\setminus\{\mathbf{0}\}} \frac{\big|b_h(\mathbf{v}_h,\mathbf{w}) - \langle \ell_h, \mathbf{w}\rangle\big|}{\big\|\mathbf{w}\big\|_{h,\mathrm{DG}}} = \sup_{\mathbf{w}\in\mathcal{V}^*\setminus\{\mathbf{0}\}} \frac{\big|b_h(\mathbf{v}_h - \mathbf{u},\mathbf{w})\rangle\big|}{\big\|\mathbf{w}\big\|_{h,\mathrm{DG}}}\,, \qquad \mathbf{v}_h \in V_h$$

cannot be computed numerically.
Inf-sup stability in the DG norm is not valid in $\mathcal{V}_h = \prod_{n,K} \mathrm{C}^1(\overline{I_{n,h}\times K};\mathbb{R}^m)$.

The practical minimal residual approach

Select an extended DG space $V_h \subset V_h^{\text{ext}} \subset \mathcal{V}_h$.

We define for all $\mathbf{v}_h \in V_h$ the solution $\mathbf{v}_h^* \in V_h^{\text{ext}}$ of

$$\left(\mathbf{v}_h^*, \mathbf{w}_h\right)_{h,\mathrm{DG}^+} = b_h\left(\mathbf{v}_h, \mathbf{w}_h\right), \qquad \mathbf{w}_h \in V_h^{\text{ext}},$$

the trial-to-test operator $\Theta_{h,\mathrm{DG}^+} \colon V_h \longrightarrow V_h^{\text{ext}}$, $\mathbf{v}_h \longmapsto \mathbf{v}_h^*$, and the dual test space

$$V_{h,\mathrm{DG}^+}^* = \left\{\Theta_{h,\mathrm{DG}^+}(\mathbf{v}_h) \colon \mathbf{v}_h \in V_h\right\} = \Theta_{h,\mathrm{DG}^+}(V_h).$$

We have $\dim V_{h,\mathrm{DG}^+}^* = \dim V_h$, and a unique Petrov-Galerkin solution $\mathbf{u}_{h,\mathrm{DG}^+} \in V_h$

$$b_h\left(\mathbf{u}_{h,\mathrm{DG}}, \mathbf{w}_h\right) = \left\langle \ell_h, \mathbf{w}_h \right\rangle, \qquad \mathbf{w}_h \in V_{h,\mathrm{DG}^+}^*.$$

exists.

The solution can be obtained by solving a symmetric positive definite Schur complement problem. The Schur complement matrix can be computed locally.

Mesh-independent inf-sup stability is an open problem.

The nonlocal practical minimal residual approach

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Now we define the trial-to-test operator $\Theta_{h,\mathrm{DG}}\colon V_h \longrightarrow V_h^{\mathrm{ext}}$ by

$$\big(\Theta_{h,\mathrm{DG}}(\mathbf{v}_h), \mathbf{w}_h\big)_{h,\mathrm{DG}} = b_h\big(\mathbf{v}_h, \mathbf{w}_h\big)\,, \qquad \mathbf{w}_h \in V_h^{\mathrm{ext}}$$

and $V_{h,\mathrm{DG}}^* = \Theta_{h,\mathrm{DG}}(V_h)$. We have $\dim V_{h,\mathrm{DG}}^* = \dim V_h$.
A unique Petrov-Galerkin solution $\mathbf{u}_{h,\mathrm{DG}} \in V_h$ exists:

$$b_h\big(\mathbf{u}_{h,\mathrm{DG}}, \mathbf{w}_h\big) = \big\langle \ell_h, \mathbf{w}_h\big\rangle\,, \qquad \mathbf{w}_h \in V_{h,\mathrm{DG}}^*\,.$$

We have inf-sup stability for $\mathbf{v}_h \in V_h$

$$c_{\mathrm{inf\text{-}sup}}^{\mathrm{ext}} \big\|\mathbf{v}_h\big\|_{h,\mathrm{DG}} \le \sup_{\mathbf{w}_h \in V_h^{\mathrm{ext}} \setminus \{\mathbf{0}\}} \frac{b_h(\mathbf{v}_h, \mathbf{w}_h)}{\big\|\mathbf{w}_h\big\|_{h,\mathrm{DG}}} = \frac{b_h(\mathbf{v}_h, \Theta_{h,\mathrm{DG}}(\mathbf{v}_h))}{\big\|\Theta_{h,\mathrm{DG}}(\mathbf{v}_h)\big\|_{h,\mathrm{DG}}}$$
$$= \sup_{\mathbf{w}_h \in V_{h,\mathrm{DG}}^* \setminus \{\mathbf{0}\}} \frac{b_h(\mathbf{v}_h, \mathbf{w}_h)}{\big\|\mathbf{w}_h\big\|_{h,\mathrm{DG}}}$$

and thus convergence.

A DPG discretization

For the special case that in $V_h^{\mathrm{tr}} \subset V_h$ the traces at the inner interfaces coincide we define the trace spaces (selecting $\mathbf{n}_F$ for $F \in \mathcal{F}_h$)

$$\widehat{V}_{n,h,\Omega} = \big\{\mathbf{v}_{n,h}(t_n) \colon \mathbf{v}_h \in V_h^{\mathrm{tr}}\big\} = \big\{\mathbf{v}_{n+1,h}(t_n) \colon \mathbf{v}_h \in V_h^{\mathrm{tr}}\big\}\,, \qquad n = 1, \dots, N-1\,,$$

$$\widehat{V}_{n,h,F} = \big\{\underline{A}_{\mathbf{n}_F}\mathbf{v}_{n,h,K}|_{I_{n,h}\times F} \colon \mathbf{v}_h \in V_h^{\mathrm{tr}}\big\}\,, \qquad F \in \mathcal{F}_K \cap \Omega\,,$$

$$\widehat{V}_h = \prod_{n=1}^{N-1} \widehat{V}_{n,h,\Omega} \times \prod_{n=1}^{N} \prod_{F\in\mathcal{F}_h\cap\Omega} \widehat{V}_{n,h,F}\,.$$

and the corresponding trace mapping $\gamma_h \colon V_h^{\mathrm{tr}} \longrightarrow \widehat{V}_h$.

For $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in V_h \times \widehat{V}_h$ and $\mathbf{w}_h \in V_h^{\mathrm{ext}}$ we define the discrete bilinear form

$$b_h^{\mathrm{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{w}_h\big) = -\big(\mathbf{v}_h, L_h\mathbf{w}_h\big)_{Q_h} - \sum_{n=1}^{N-1}\big(M_h\widehat{\mathbf{v}}_h(t_n), [\mathbf{w}_h]_n\big)_\Omega + \big(M_h\mathbf{v}_h(t_N), \mathbf{w}_h(t_N)\big)_\Omega$$
$$+ \sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big(\widehat{\mathbf{v}}_h, \underline{A}^{\mathrm{up}}_{\mathbf{n}_K}[\mathbf{w}_h]_{K,F}\big)_{I_h\times F} + \sum_{F\in\mathcal{F}_h\cap\partial\Omega}\big(\mathbf{v}_h, \underline{A}^{\mathrm{up}}_{\mathbf{n}_K}[\mathbf{w}_h]_{K,F}\big)_{I_h\times F}$$

We assume $\big\{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in V_h \times \widehat{V}_h \colon b_h^{\mathrm{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{w}_h\big) = 0 \text{ for all } \mathbf{w}_h \in V_h^{\mathrm{ext}}\big\} = \{(\mathbf{0}, \mathbf{0})\}$.

A DPG method

We define the norms

$$\big\|(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}} = \Big(h\,\big\|M^{-1/2}L_h\mathbf{v}_h\big\|^2_{Q_h} + \sum_{n=1}^{N-1}\big\|M_h^{1/2}\widehat{\mathbf{v}}_h(t_n)\big\|^2_\Omega + \big\|M_h^{1/2}\mathbf{v}_h(t_N)\big\|^2_\Omega$$

$$+ \sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big\|\widehat{\mathbf{v}}_h\big\|^2_{I_h\times F} + \sum_{F\in\mathcal{F}_h\cap\partial\Omega}\big\|\underline{A}^{\mathrm{up}}_{\mathbf{n}_K}\mathbf{v}_h\big\|^2_{I_h\times F}\Big)^{1/2},$$

$$\big\|\mathbf{w}_h\big\|_{h,\mathrm{DPG}^+} = \Big(h^{-1}\big\|M^{1/2}\mathbf{w}_h\big\|^2_{Q_h} + \sum_{n=1}^{N}\Big(\big\|M_h^{1/2}w_{n,h}(t_{n-1})\big\|^2_\Omega + \big\|M_h^{1/2}w_{n,h}(t_n)\big\|^2_\Omega\Big)$$

$$+ \sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K}\big\|\underline{A}_{\mathbf{n}_K}\mathbf{w}_h\big\|^2_{I_h\times F}\Big)^{1/2}.$$

We have continuity $\big|b_h^{\mathrm{DPG}}\big((\mathbf{v}_h,\widehat{\mathbf{v}}_h),\mathbf{w}_h\big)\big| \le \big\|(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}}\big\|\mathbf{w}_h\big\|_{h,\mathrm{DPG}^+}$.

Now we define the trial-to-test operator $\Theta_{h,\mathrm{DPG}^+}\colon V_h\times\widehat{V}_h \longrightarrow V_h^{\mathrm{ext}}$ by

$$\big(\Theta_{h,\mathrm{DPG}^+}(\mathbf{v}_h,\widehat{\mathbf{v}}_h),\mathbf{w}_h\big)_{h,\mathrm{DPG}^+} = b_h^{\mathrm{DPG}}\big((\mathbf{v}_h,\widehat{\mathbf{v}}_h),\mathbf{w}_h\big)\,, \qquad \mathbf{w}_h\in V_h^{\mathrm{ext}}$$

and $V^*_{h,\mathrm{DPG}^+} = \Theta_{h,\mathrm{DPG}^+}(V_h)$.
We have $\big\|\Theta_{h,\mathrm{DPG}^+}(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}^+} \le \big\|(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}}$.

Inf-sup stability of the DPG method

Assume that $\mathbf{y}_h \in V_h^{\text{ext}}$ can be constructed such that

$$\big\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}}^2 \le C_1\, b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{y}_h\big)$$

and

$$\big\|\mathbf{y}_h\big\|_{h,\mathrm{DPG+}} \le C_2\, \big\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}}\,.$$

Then, we have

$$\begin{aligned}
\big\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}}^2 &\le C_1\, b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{z}_h\big) \le C_1\, \big\|\mathbf{y}_h\big\|_{h,\mathrm{DPG+}} \frac{b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{y}_h\big)}{\big\|\mathbf{y}_h\big\|_{h,\mathrm{DPG+}}} \\
&\le C_1 C_2\, \big\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}} \frac{b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{y}_h\big)}{\big\|\mathbf{y}_h\big\|_{h,\mathrm{DPG+}}}\,.
\end{aligned}$$

This yields inf-sup stability in $(V_h \times \widehat{V}_h) \times V_h^{\text{ext}}$ and also in in $(V_h \times \widehat{V}_h) \times V_{h,\mathrm{DPG+}}^*$ by

$$\begin{aligned}
\frac{1}{C_1 C_2}\big\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG}} &\le \frac{b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{y}_h\big)}{\big\|\mathbf{y}_h\big\|_{h,\mathrm{DPG+}}} \le \sup_{\mathbf{w}_h \in V_h^{\text{ext}} \setminus \{\mathbf{0}\}} \frac{b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{w}_h\big)}{\big\|\mathbf{w}_h\big\|_{h,\mathrm{DPG+}}} \\
&= \frac{b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \Theta_{h,\mathrm{DPG+}}(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big)}{\big\|\Theta_{h,\mathrm{DPG+}}(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\big\|_{h,\mathrm{DPG+}}} = \sup_{\mathbf{w}_h \in V_{h,\mathrm{DPG+}}^* \setminus \{\mathbf{0}\}} \frac{b_h^{\text{DPG}}\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), \mathbf{w}_h\big)}{\big\|\mathbf{w}_h\big\|_{h,\mathrm{DPG+}}}\,.
\end{aligned}$$

Summary and Outlook

We aim to transfer the error estimate for the DG solution with full upwind

$$\left\|\mathbf{u}-\mathbf{u}_h\right\|_{h,\mathrm{DG}} \leq C h^{s-1/2} \left\|\mathrm{D}^s \mathbf{u}\right\|_Q + C T\, h^{-1/2} \left\|M_h^{-1/2}(M_h - M)\partial_t \mathbf{u}\right\|_Q$$

to the DPG method.

- The constants depend on the inverse estimates and thus on the polynomial degree in V_h; thus, we get larger constants in V_h^{ext}.
- Both solutions can be obtained by a symmetric positive definite system.
- It has to be checked numerically, which method is more efficient.

Di Pietro / Ern: Mathematical aspects of discontinuous Galerkin methods. Springer 2011

Ern / Guermond: Finite elements III: First-Order and Time-Dependent PDEs. Springer 2021

Bansal / Moiola / Perugia / Schwab: Space–time discontinuous Galerkin approximation of acoustic waves with point singularities. IMA J. Num. Anal. 2021

Dörfler / Wieners: Space-time approximations for linear acoustic, elastic, and electro-magnetic wave equations. Lecture Notes for the MFO seminar on wave phenomena 2019

Corallo / Dörfler / Wieners: Space-time discontinuous Galerkin methods for weak solutions of hyperbolic linear symmetric Friedrichs systems. Preprint 2022