DPG Progress at Oden

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Minimum Residual & Least-Squares Finite Element Methods
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The University of Texas at Austin
Oden Institute for Computational Engineering and Sciences

Outline

1. Full Envelope Approximation for Waveguide Problems
2. Double Adaptivity
3. Automatic hp-Adaptivity with DPG

Analysis of Full Envelope UW Formulation for Linear Waveguide Problem

UW formulation

$$\left\{\begin{array}{l} u \in D(A) \\ Au = f \end{array}\right. \Rightarrow \left\{\begin{array}{l} u \in L^2(\Omega) \\ (u, A^*v) = (f, v)\; v \in D(A^*) \end{array}\right. \Rightarrow \left\{\begin{array}{l} u \in L^2(\Omega),\; \hat{u} \in \hat{U} \\ (u, A^*v) + \langle \hat{u}, v\rangle = (f, v)\; v \in H_{A^*}(\Omega_h) \end{array}\right.$$

Inf–sup constant γ depends upon boundedness below constant α and scaling parameter β in the adjoint graph norm

$$\left.\begin{array}{l} \alpha\|u\| \leq \|Au\|,\; u \in D(A) \\ \|v\|_V^2 := \|A^*v\|^2 + \beta^2\|v\|^2 \end{array}\right\} \quad \Rightarrow \quad \gamma \geq [1 + (\frac{\beta}{\alpha})^2]^{-1/2} .$$

(Ideal) DPG reproduces the stability of the continuous problem

$$\underbrace{\|u - u_h\|^2}_{L^2-\text{error}} \leq \underbrace{[1 + (\frac{\beta}{\alpha})^2]}_{\text{stability constant}} \{\underbrace{\inf_{w_h \in U_h} \|u - w_h\|^2}_{\text{field BA error}} + \underbrace{\inf_{\hat{w}_h \in \hat{U}_h} \|\hat{u} - \hat{w}_h\|^2}_{\text{trace BA error}}\}$$

Analysis of Full Envelope UW Formulation for Linear Waveguide Problem

Def. Full envelope operator

$$\tilde{A}\tilde{u} := e^{ikz} A(e^{-ikz}\tilde{u})$$

Thm. Full envelope operator inherits boundedness below constant from the original operator

$$\|Au\| \geq \alpha \|u\| \quad \Leftrightarrow \quad \|\tilde{A}\tilde{u}\| \geq \alpha \|\tilde{u}\|$$

Proof:

$$\|\tilde{A}\tilde{u}\| = \|e^{ikz} A(e^{-ikz}\tilde{u})\| = \|A(e^{-ikz}\tilde{u})\| \geq \alpha \|e^{-ikz}\tilde{u}\| = \alpha \|\tilde{u}\|$$

Thm. Inverse of the boundedness below constant depends linearly upon waveguide length L (a technical result)

$$\|Au\| \geq \underbrace{\frac{\alpha_0}{L}}_{=:\alpha} \|u\|$$

Stability of the strong formulation, Sketch of the proof.

$$D \subset \mathbb{R}, \quad L > 0, \quad \Omega := D \times (0, L), \quad a = a(x) \in GL(\mathbb{R}^{2\times 2}), a > 0\,.$$

Acoustics:

$$\begin{aligned} i\omega u + a\nabla p &= f && \text{in } \Omega \\ i\omega p + \nabla \cdot u &= g && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma_i := D \times \{0\} \\ u \cdot n &= 0 && \text{on } \Gamma_l := \partial D \times (0, L) \\ i\omega u \cdot n - DtNp &= 0 && \text{on } \Gamma_o := D \times \{l\}\,. \end{aligned}$$

Eigenvalue problem:

$$\begin{cases} -\nabla \cdot (a\nabla\phi_n) &= \lambda_n^2 \phi_n \quad \text{in } D \\ n \cdot a\nabla\phi_n &= 0 \quad \text{on } \partial D\,. \end{cases}$$

$$(\phi_n, \phi_m) = \delta_{nm} \qquad (a\nabla\phi_n, \nabla\phi_m) = \delta_{nm}\lambda_n^2\,.$$

Lemma Solution $p \in H^1(\Omega)$, $p = 0$ on Γ_i of

$$(a\nabla p, \nabla q) - \omega^2(p, q) + \langle DtNp, q\rangle = (f, \nabla q) \quad q \in H^1(\Omega),\ q = 0 \text{ on } \Gamma_i$$

satisfies

$$\|p\|_{H^1(\Omega)} \le CL\|f\|\,.$$

Stability of the strong formulation. Sketch of the proof.

Proof: Use decompositions

$$p = \sum_i \phi_i(x)\psi_i(z), \quad f = \sum_i (f,\phi_i)\phi_i$$

and reduce the stability analysis to 1D problems for $\psi_i(z)$.
Electromagnetics (in progress): $(\mu = 1, \epsilon = 1 + \Delta\epsilon)$

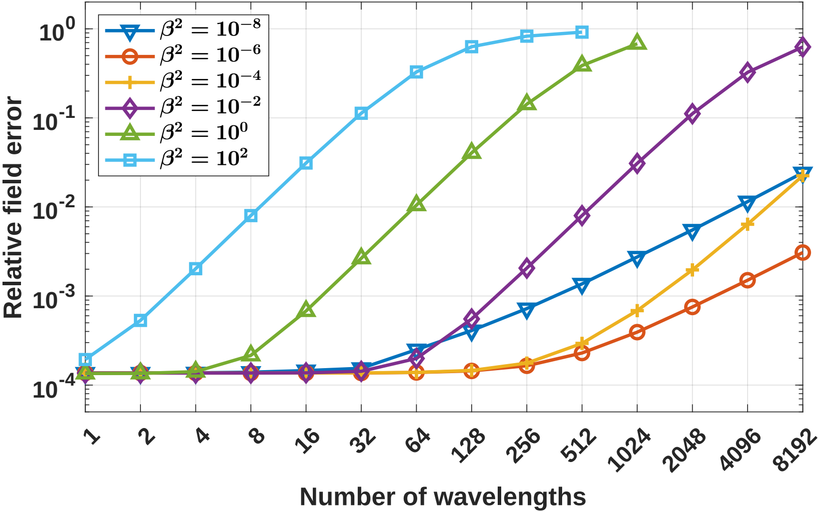
$$\begin{array}{rll} \nabla \times E & = -i\omega H & \text{in } \Omega \\ \nabla \times H & = i\omega\epsilon E & \text{in } \Omega \\ n \times E & = 0 & \text{on } \Gamma_l \\ n \times E & = 0 & \text{on } \Gamma_i \\ n \times E + DtN H \cdot n & = 0 & \text{on } \Gamma_0 \,. \end{array}$$

Eigenvalue problem for propagation constant β:

$$\left\{ \begin{array}{l} E \in H_0(\text{curl}, D) \cap H_0(\text{div}, D) \\ \nabla \times \text{curl} E - \omega^2 \epsilon E - \nabla(\frac{1}{\epsilon}\text{div}\epsilon E) = -\beta^2 E_t \end{array} \right.$$

is neither self-adjoint in $L^2(D)$ nor in $L^2_\epsilon(D)$ but it is a $\Delta\epsilon$-perturbation of a self-adjoint problem in $L^2(D)$.

Positive Effect of Small β on Pollution



Pollution error in a 3D rectangular waveguide for ultraweak DPG Maxwell with test norm

$$\|v\|^2_{\mathcal{V}(\Omega_h)} = \|\nabla_h \times \boldsymbol{F} - (i\omega\varepsilon + \sigma)^* \boldsymbol{G}\|^2 + \|\nabla_h \times \boldsymbol{G} + (i\omega\mu)^* \boldsymbol{F}\|^2 + \beta^2 \left(\|\boldsymbol{F}\|^2 + \|\boldsymbol{G}\|^2\right).$$

Outline

Three Mixed Problems

A variational problem reformulated as a mixed problem

$$\left\{\begin{array}{lll} \psi \in V\,, u \in U & & \\ (\psi, v)_V + b(u,v) & = l(v) & v \in V \\ b(\delta u, \psi) & = 0 & \delta u \in U\,. \end{array}\right. \tag{1}$$

Ideal PG method with optimal test functions

$$\left\{\begin{array}{lll} \psi^h \in V\,, \tilde{u}_h \in U_h & & \\ (\psi^h, v)_V + b(\tilde{u}_h,v) & = l(v) & v \in V \\ b(\delta u_h, \psi^h) & = 0 & \delta u_h \in U_h\,. \end{array}\right. \tag{2}$$

Practical PG method with optimal test functions

$$\left\{\begin{array}{lll} \psi_h \in V_h\,, u_h \in U_h & & \\ (\psi_h, v_h)_V + b(u_h,v_h) & = l(v_h) & v_h \in V_h \\ b(\delta u_h, \psi_h) & = 0 & \delta u_h \in U_h\,. \end{array}\right. \tag{3}$$

The Double Adaptivity Algorithm of Dahmen et al. [1]

```
Set initial trial mesh U_h
do !  outer adaptive loop
  (re)set test mesh V_h to coincide with trial mesh U_h
  do !  inner adaptive loop
    solve (3) on the current trial and test meshes
    estimate error ||ψ^h − ψ_h||_V ≤ err_V and compute norm ||ψ_h||_V
    if err_V/||ψ_h||_V < tol_V exit the inner (test) loop
    adapt test mesh V_h using element contributions of err_V
  enddo
  compute trial norm of the solution ||u_h||_U
  if ||ψ_h||_V/||u_h||_U < tol_U STOP
  use element contributions to ||ψ_h||_V to refine the trial mesh
enddo
```

[1] A. Cohen, W. Dahmen, and G. Welper. "Adaptivity and Variational Stabilization for Convection-Diffusion Equations". In: *ESAIM Math. Model. Numer. Anal.* 46.5 (2012), pp. 1247–1273

Ultraweak Variational Formulation of the Confusion Problem

Confusion problem:

$$
\begin{aligned}
&\mathsf{u} := (\sigma, u) \in D(A) := H(\mathrm{div}, \Omega) \times H^1_0(\Omega) \subset (L^2(\Omega))^N \times L^2(\Omega)\\
&A \, : \, D(A) \to (L^2(\Omega))^N \times L^2(\Omega)\\
&A\mathsf{u} = A(\sigma, u) := (\sigma - \epsilon^{\frac{1}{2}} \nabla u, -\epsilon^{\frac{1}{2}} \mathrm{div}\, \sigma + \beta \cdot \nabla u)\,.
\end{aligned}
$$

Closed operator setting:

$$
\begin{aligned}
&\mathsf{u} := (\sigma, u) \in D(A) := H(\mathrm{div}, \Omega) \times H^1_0(\Omega) \subset (L^2(\Omega))^N \times L^2(\Omega)\\
&A \, : \, D(A) \to (L^2(\Omega))^N \times L^2(\Omega)\\
&A\mathsf{u} = A(\sigma, u) := (\sigma - \epsilon^{\frac{1}{2}} \nabla u, -\epsilon^{\frac{1}{2}} \mathrm{div}\, \sigma + \beta \cdot \nabla u)\,.
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{v} := (\tau, v) \in D(A^*) = D(A) \subset (L^2(\Omega))^N \times L^2(\Omega)\\
&A^* \, : \, D(A^*) \to (L^2(\Omega))^N \times L^2(\Omega)\,,\\
&A^*\mathsf{v} = A^*(\tau, v) = (\tau + \epsilon^{\frac{1}{2}} \nabla v, \epsilon^{\frac{1}{2}} \mathrm{div}\, \tau - \mathrm{div}\,(\beta v))
\end{aligned}
$$

UW formulation:

$$
\left\{
\begin{array}{l}
\mathsf{u} \in D(A)\\
A\mathsf{u} = \mathsf{f}
\end{array}
\right.
\quad \Leftrightarrow \quad
\left\{
\begin{array}{ll}
\mathsf{u} \in (L^2(\Omega))^N \times L^2(\Omega) & \\
(\mathsf{u}, A^*\mathsf{v}) = (\mathsf{f}, \mathsf{v}) & \mathsf{v} \in D(A^*)\,.
\end{array}
\right.
\tag{4}
$$

where $D(A^*)$ is equipped with the adjoint graph norm:

$$
\|\mathsf{v}\|^2_V = \|A^*\mathsf{v}\|^2 + \alpha \|\mathsf{v}\|^2
$$

A-Posteriori Error Estimate Based on the Duality Theory [2]

The semidiscrete version of the UW problem (4) is equiavlent to the *primal minimization problem:*

$$\inf_{\substack{\psi \in D(A^*) \\ A^*\psi \in U_h^\perp}} \underbrace{\frac{1}{2}\|A^*\psi\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f},\psi)}_{=:J(\psi)} \tag{5}$$

where $U_h^\perp$ denotes the $L^2(\Omega)$-orthogonal complement of trial space U_h.
The minimization problem (5) is equivalent to the *dual maximization problem:*

$$(^{**}) = \sup_{\phi\in D(A)} \underbrace{-\frac{1}{2}\|\phi^\perp\|^2 - \frac{1}{2\alpha}\|\mathsf{f} - A\phi\|^2}_{=:J^*(\phi)} = -\inf_{\phi\in D(A)} \underbrace{\frac{1}{2}\|\phi^\perp\|^2 + \frac{1}{2\alpha}\|\mathsf{f} - A\phi\|^2}_{=-J^*(\phi)} . \tag{6}$$

The energy error for the solutions to the fully discrete primal and dual problems is estimated by the discrete *duality gap*:

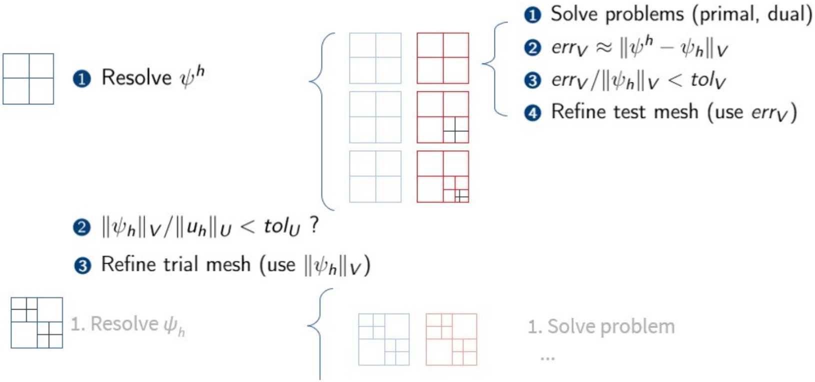
$$\left.\begin{array}{r} \|A^*(\psi-\psi_h)\|^2 + \alpha\|\psi-\psi_h\|^2 \\ \|\phi^\perp - \phi_h^\perp\|^2 + \frac{1}{\alpha}\|A(\phi-\phi_h)\|^2 \end{array}\right\} \le 2(J(\psi_h) - J^*(\phi_h)) . \tag{7}$$

Local contributions to the duality gap estimate serve as error indicators for the inner adaptivity loop:

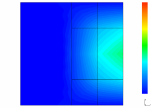
$$2(J(\psi_h) - J^*(\phi_h)) = \frac{1}{\alpha}\int_\Omega \alpha(A^*\psi_h - \phi_h^\perp)^2 + (\alpha\psi_h - (\mathsf{f} - A\phi_h))^2 . \tag{8}$$

[2] L. Demkowicz et al. “The Double Adaptivity Paradigm (How to circumvent the discrete inf-sup conditions of Babuška and Brezzi)”. In: *Comput. Math. Appl.* 95 (2021), pp. 41–66. DOI: https://doi.org/10.1016/j.camwa.2020.10.002

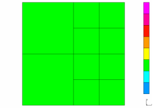
Double adaptivity



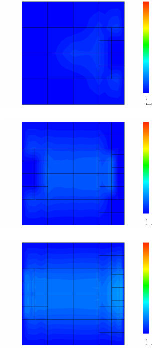
Test mesh refinement



Refined trial mesh



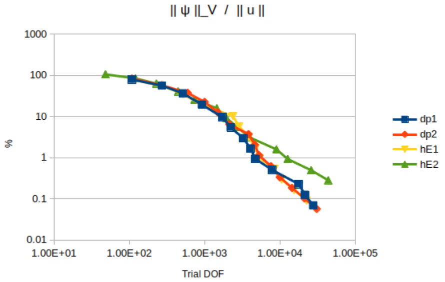
Reset test mesh



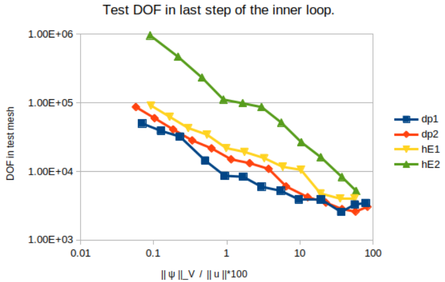
Final test mesh (second outer loop step)

Figure: Second outer loop step and its corresponding inner loop which takes 3 steps to reach the tolerance tol_V. Trial mesh: solution u. Test mesh: H^1-component of ψ_h. $p = 3$ and $\varepsilon = 10^{-2}$.

Numerical Experiments [3]. Enrichment Strategies



Outer loop, convergence in the Riesz representation of the residual.



Inner loop total DOF vs Riesz representation of the residual.

Figure: Differences in enrichment strategies: dp increases the polymonial degree of the test functions, while hE applies (1 or 2) global refinements to the test mesh.

The convergence is very similar, however the cost of each test mesh adaptivity is very different. hE strategy result in much larger number of DOF.

[3] J. Salazar and L. Demkowicz. *The Double Adaptivity Paradigm: Conforming vs. Weakly Conforming Test Functions*. Tech. rep. 15. Oden Institute for Computational Engineering and Sciences, 2021

Adaptive refinement strategies and parameters

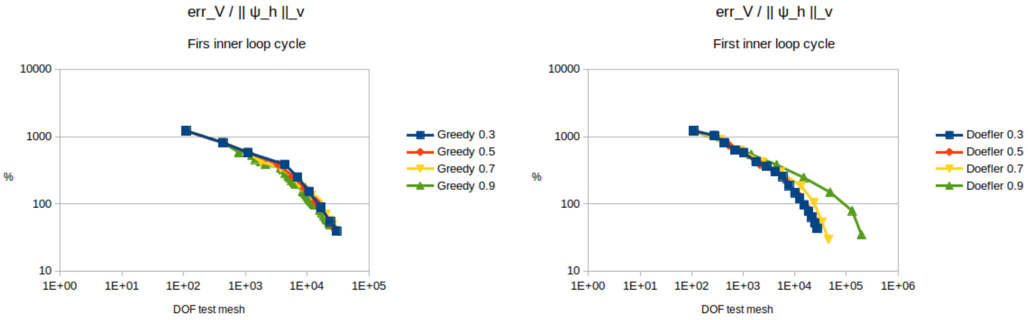


Figure: Comparison of Doefler and Greedy refinement strategies and parameters.

Although similar convergence behaviour, for greedy, a threshold of 0.3 performs fewer iterations; for Doefler a threshold of 0.9 performs fewer iterations.

Refinement factor 0.7, 0.5, 0.3 with Doerfler Strategy.

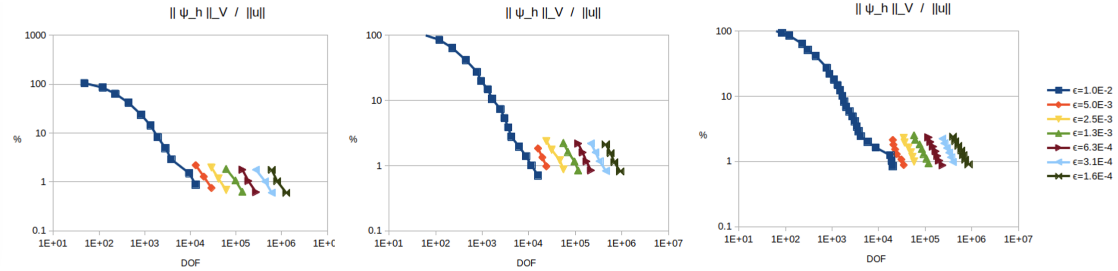


Figure: Outer loop convergence comparison between different refinement factors using the continuation strategy. Left: 0.7, center: 0.5, right: 0.3.

Using different refinement factors result in similar convergence, however the number of iterations required change considerably. A factor of 0.7 performed the best.

Parameter α

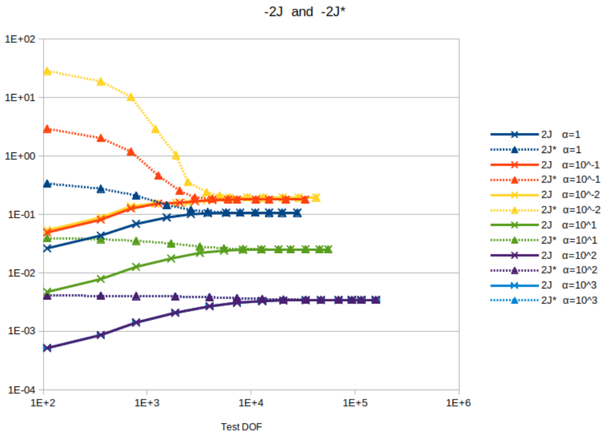


Figure: Primal and dual energies for multiple values of α, $\varepsilon = 10^{-2}$.

Convergence of the duality gap for different values of α, since it appears multiplying for the primal energy and dividing for the dual energy we observe that large values of α impact the dual energy the most, while small values of α impact the primal the most.

Parameter α

A successful error estimator η must satisfy the efficiency and reliability properties:

$$\eta \le C_1 \|e\|_E \quad \text{and} \quad \|e\|_E \le C_2 \eta \ .$$

From (7) we verify reliability. For efficiency, we verify numerically via the effectivity index $\theta = \eta / \|e\|_E$:

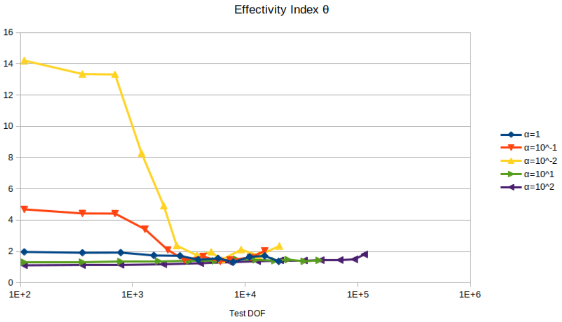


Figure: Effectivity index $\theta = \eta / \|e\|_E$, $\varepsilon = 10^{-2}$.

Convergence of ψ_h in trial mesh, low order elements.

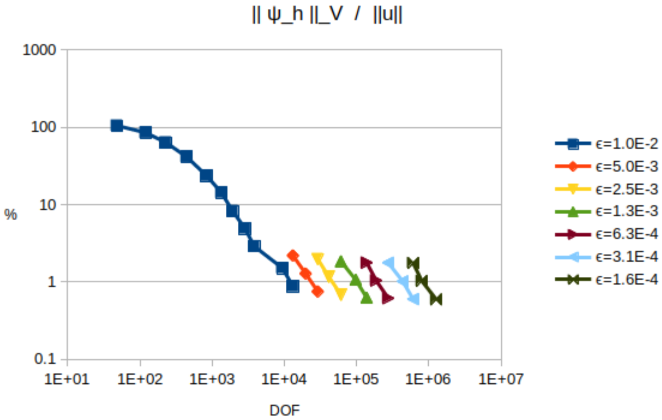


Figure: Convergence using the continuation technique on ϵ, with a refinement factor of 0.7 for the trial mesh, $p_U = 2$, $p_V = 3$.

Convergence of ψ_h in trial mesh, high order elements.

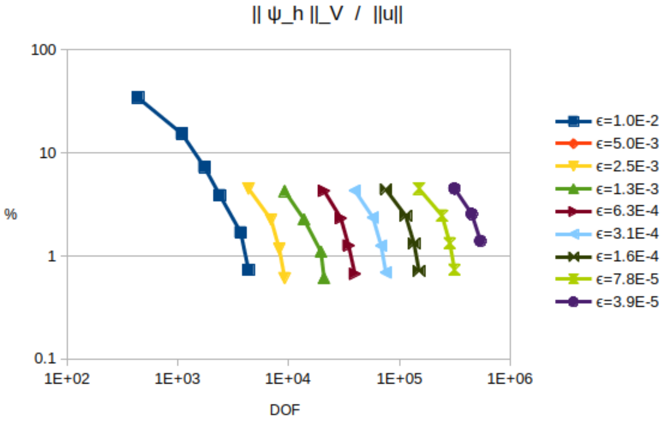


Figure: Convergence using the continuation technique on ϵ, with a refinement factor of 0.7 for the trial mesh, $p_U = 6$, $p_V = 7$.

Parameter α

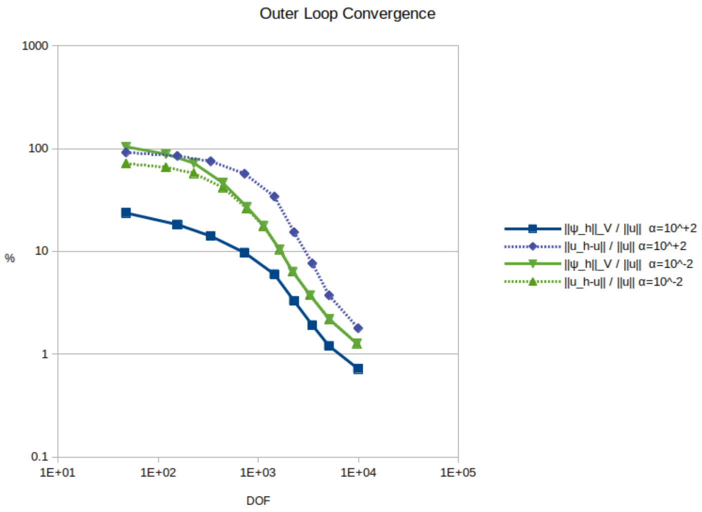


Figure: Impact of coefficient α in the convergence of $\|\psi_h\|_V$ and $\|u - u_h\|$, $\varepsilon = 10^{-3}$.

Small values of α result in $\|\psi_h\|_V$ being a better approximation of $\|u - u_h\|$.

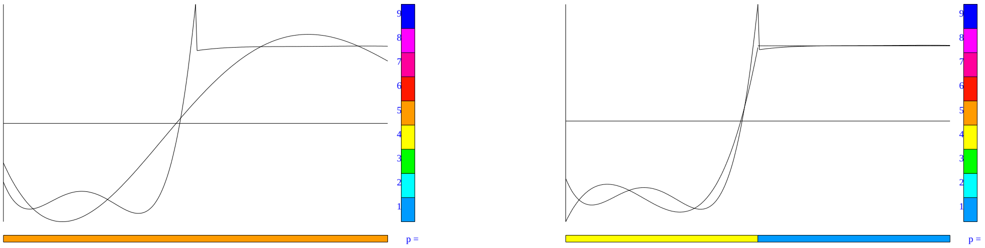
Outline

Automatic hp-Adaptivity Algorithm [4]

```
Set initial trial mesh U_h
do ! adaptive loop
  solve the problem on the current mesh
  use the element residuals to mark elements for refinement
  dumpout the current mesh
  hp-refine the marked elements
  solve the problem on the hp-refined mesh
  for each refined element K
    determine the error decrease rate for p- and h-refinements
    use the rates to decide between p- and h-refinement
    update the maximum error decrease rate max_K(Δe_K)
  endfor
  dumpin the last mesh
  for each element marked for refinement
    if the element optimal error decrease rate Δe_K > 0.25 max_K(Δe_K) then
      refine the element in the optimal way
    endif
  endfor
enddo
```

[4] L. Demkowicz. *Computing with hp Finite Elements. I.One- and Two-Dimensional Elliptic and Maxwell Problems.* Boca Raton: Chapman & Hall/CRC Press, Taylor and Francis, 2006

Error Decrease Rate as the Criterion to Choose Between p- and h-Refinement



p-refinement:

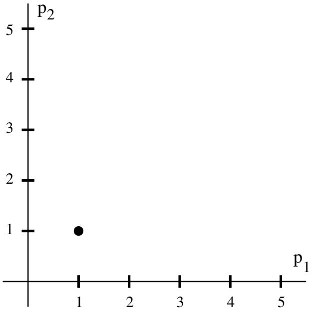
$$\begin{aligned}\Delta e &:= \frac{\|u_{h/2,p+1}-u_{h,p}\|^2-\|u_{h/2,p+1}-u_{h,p+1}\|^2}{1}\\ &= 0.27609-0.25657=0.01952\end{aligned}$$

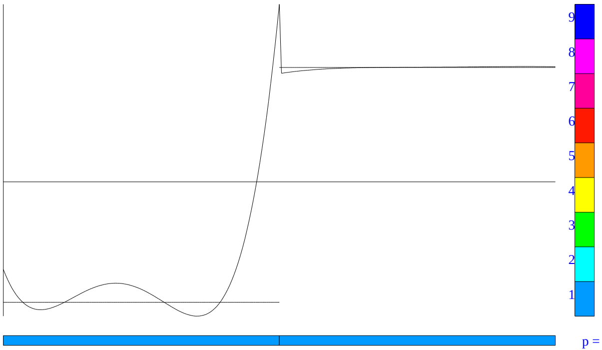
Competing h-refinement:

$$\begin{aligned}\Delta e &:= \frac{\|u_{h/2,p+1}-u_{h,p}\|^2-\|u_{h/2,p+1}-u_{h,p+1}\|^2}{\Delta N}\\ &= 0.27609-0.04413=0.23196\end{aligned}$$

The h-refinement wins.

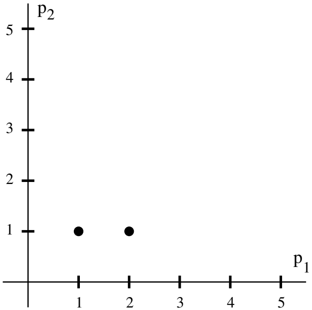
Maximum error decrease path to find the competing h-refinement

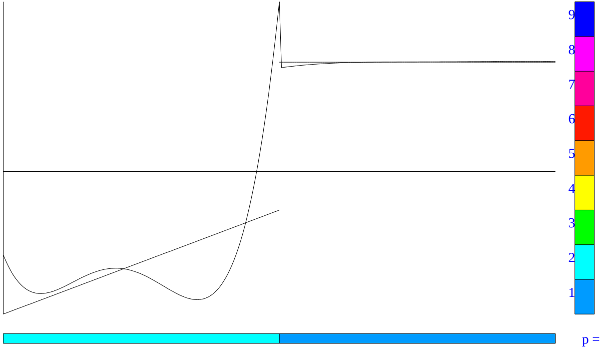




Error = 0.375048, 0.000236

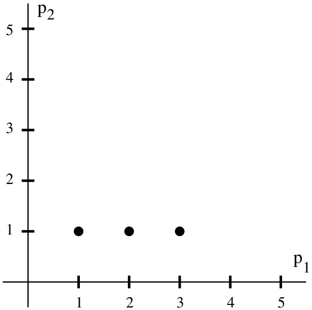
Maximum error decrease path

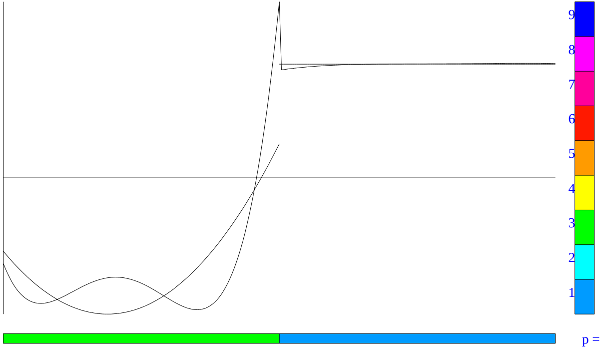




Error = 0.224857, 0.000236

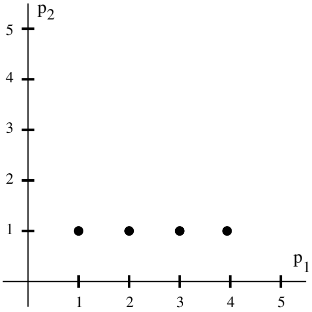
Maximum error decrease path

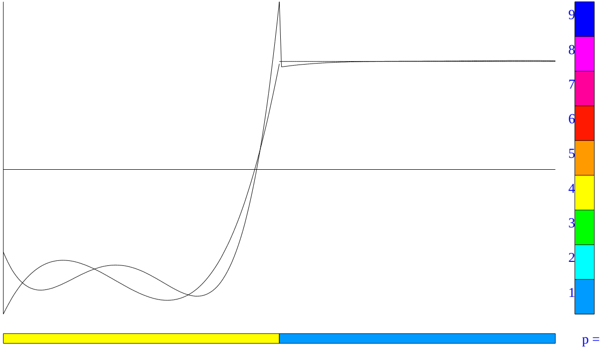




Error = 0.123835, 0.000236

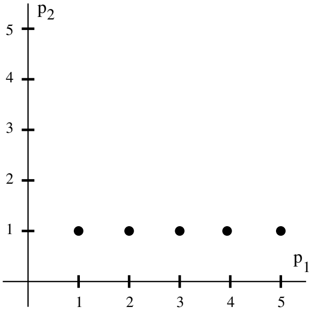
Maximum error decrease path - the competing h-refinement

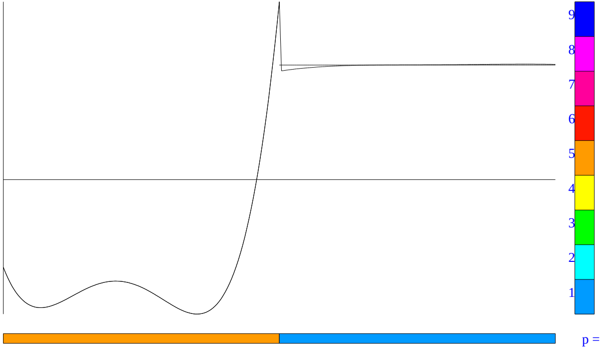




Error = 0.043898, 0.000236

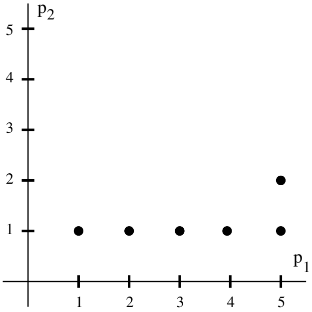
Maximum error decrease path

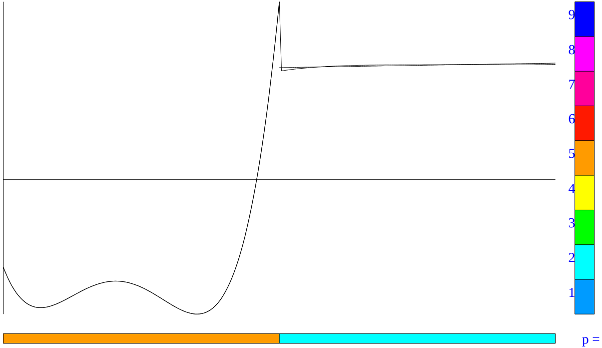




Error = 0.0, 0.000236

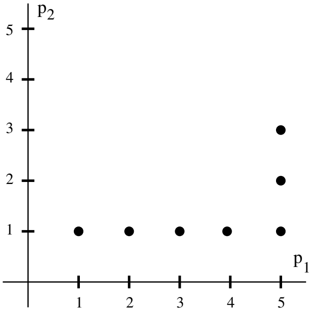
Maximum error decrease path

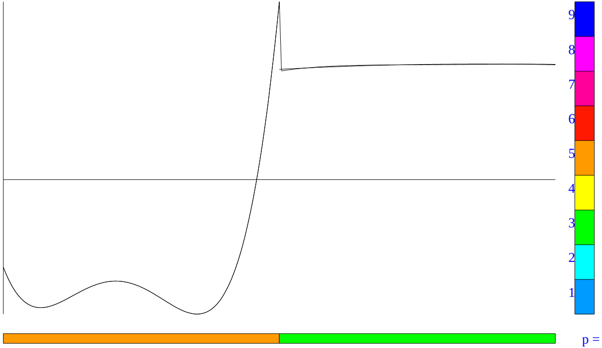




Error $= 0.0,\ 0.000076$

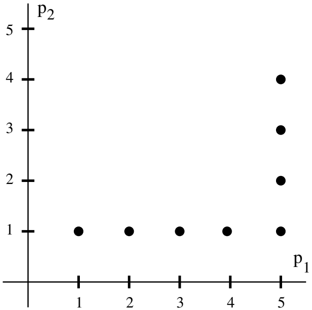
Maximum error decrease path

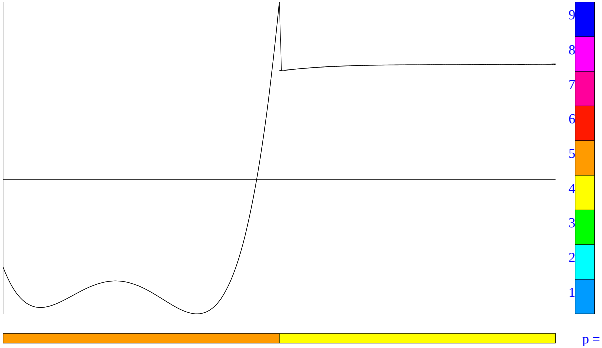




Error $= 0.0, 0.000022$

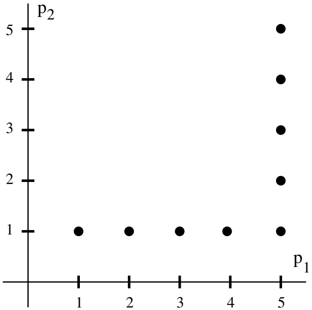
Maximum error decrease path

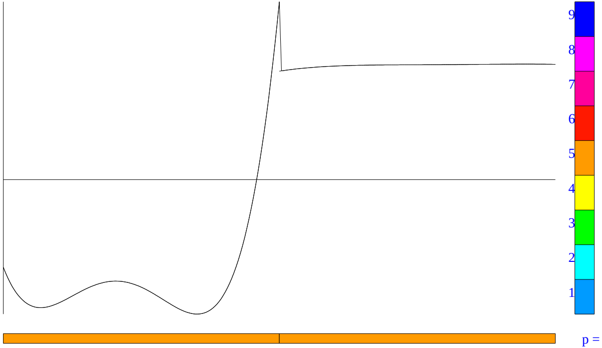




Error = 0.0, 0.000004

Maximum error decrease path



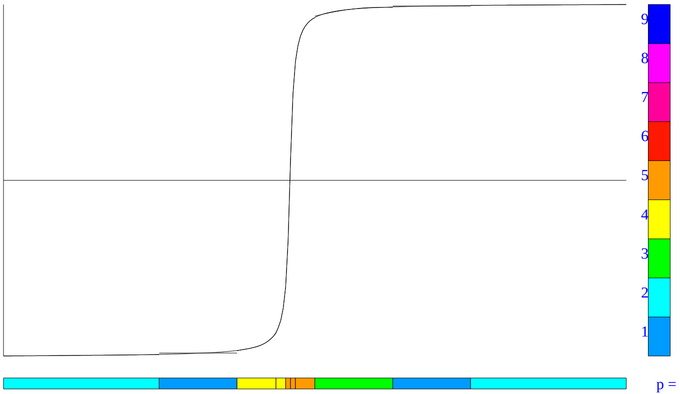


Error = 0.0, 0.0

Optimal element refinement

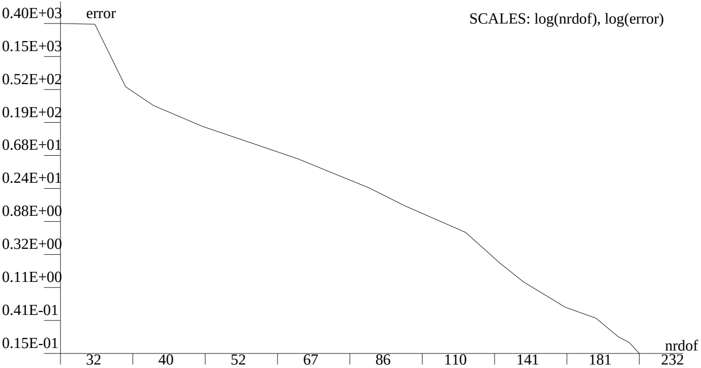
```
For each element marked for refinement
  If $\Delta e_K > 0.25\,\max_K(\Delta e_K)$ then
    if $\Delta e_{K,p} > \Delta e_{K,h}$ then
      $p$-refine the element
    else
      select the maximum $h$-refinement for which the $\Delta e_{K,h} > 0.25\,\max_K(\Delta e_K)$
    endif
  endif
endfor
```

Poisson Problem. Solution with Internal Layer



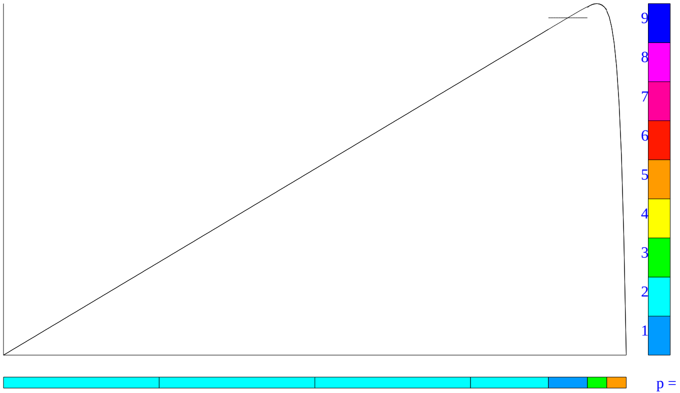
Exact and numerical solutions with the optimal mesh, error = .67 percent.

Poisson Problem. Solution with Internal Layer



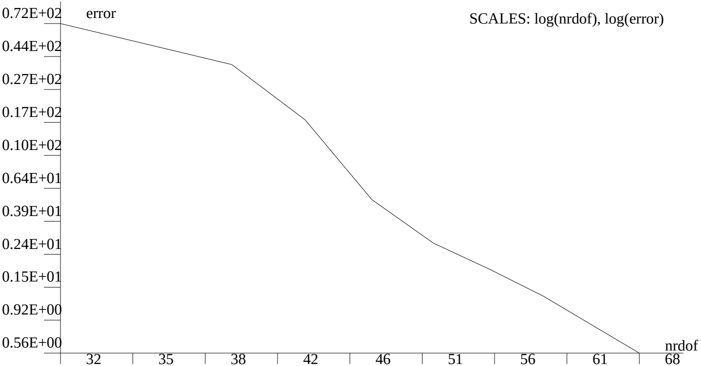
Convergence history.

Confusion Problem.



Exact and numerical solutions with the optimal mesh, error = .67 percent.

Confusion Problem



Convergence history.

Acknowledgements

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Thank you for your attention!

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